Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 35, 2021

ON FACTORIZATION OF MONOIDS

Tamar Mesablishvili

Abstract. We prove that there is a one-to-one correspondence between the set of factorizations of a monoid and the set of certain pairs consisting of a left and a right congruence on the monoid.

Keywords and phrases: Monoid, factorization, congruence.

AMS subject classification (2010): 18C05, 18C15, 18D35, 20M50.

We use [1] as a reference for the theory of monoids and their actions

1 Factorization of monoids. A *monoid* is a triple $(A, m, 1)$ consisting of a set A, an associative binary operation m, and a two-sided unit element $1 \in A$. We will often follow the common practice of writing A instead of $(A, m_A, 1_A)$. A *submonoid* of a monoid A is a subset B of A that is closed under the monoid operation and contains the identity element 1 of A. If X is a submonoid of A, ι_X denotes the canonical inclusion $X \to A$.

A monoid A is said to be *factorizable* if it contains two submonoids A_1 and A_2 such that the multiplication map $A_1 \times A_2 \to A$, $(a_1, a_2) \longmapsto a_1 a_2$ is bijective. The couple (A_1, A_2) is called a *factorization* of A. We write $\mathsf{FAC}(A)$ the set of factorizations of A.

2 Monoid actions. Let $A = (A, m, 1)$ be a monoid. A *left A-set* is a pair (X, ρ_X) consisting of a set X and a map $\rho_X : A \times X \to X$ written as $\rho_X(a, x) = ax$ and called the *action* of A on X, such that

$$a(a'x) = (aa')x, \;\; 1x = x \;\text{ for all }\; a, a' \in A, x \in X.$$

The monoid A is said to *act on X* (from the left). The set X is called a *(left) A-set*. A morphism from a left A-set X to a left A-set Y is a map $f : X \to Y$ such that

$$f(ax) = af(x) \;\text{ for all }\; a \in A, x \in X.$$

Morphisms of left A-sets are sometimes called *A-morphisms*. Right A-sets and their morphisms are defined symmetrically.

3 Congruences on an action. Let A be a monoid and let X be a left A-set. A *congruence* on a left A-set X is an equivalence relation $\wp \subseteq X \times X$ on X such that $(x, x') \in \wp$ implies $(ax, ax') \in \wp$ for all $x, x' \in X$ and $a \in A$. (Similarly we can define a *right A-congruence* on a right A-set Y.) The $\wp$-equivalence class of an element $x \in X$ is denoted by $[x]_\wp$. The set $X/\wp = \{[x]_\wp : x \in X\}$ of equivalence classes is a left A-set under the left A-action defined by $a \cdot [x]_\wp = [ax]_\wp$ and the canonical map $\pi^\wp : X \to X/\wp$

that sends every element x to its equivalence class $[x]_{\wp}$ is a morphism of left A-sets. $\pi^{\wp}$ called the *quotient map associated to the congruence* $\wp$. The set of all congruences on a left A-set X is denoted by $\mathcal{C}_A(X)$.

A *left* (resp. *right*) *congruence* on the monoid A is a congruence on the left (resp. right) A-set (A, m). A *congruence* on A is an equivalence relation on A that is both a right and a left congruence. We denote the set of all congruences on A by $Con(A)$.

Congruences typically arise as kernels of morphisms: For any morphism $f : X \to Y$ of left (resp. right) A-sets, the subset

$$\mathsf{K}[f] = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$$

of the set $X \times X$ is a congruence on the left (resp. right) A-set X and is called the *kernel congruence* of f. In fact every congruence is the kernel congruence of some A-morphism, namely the quotient map associated to the congruence. Explicitly, if $\wp$ is a congruence on a (left or right) A-set X, then $\mathsf{K}[\pi^{\wp}] = \wp$.

Given a left A-set X, there are always two special congruences on X: The *trivial* (or, *diagonal*) congruence $\triangle_X$ (which identifies elements with themselves only) and the full congruence ∇_X (which identifies together all elements). Thus,

$$\triangle_X = \{(x, x) : x \in X\} \text{ and } \nabla_X = X \times X.$$

In the special case of the left (resp. right) A-set (A, m), we write $\mathcal{C}^l(A)$ (resp. $\mathcal{C}^r(A)$) for the corresponding set of left (resp. right) congruences on it. If we consider the natural order in $\mathcal{C}_A(X)$ given by inclusion, then ∇_X is the greatest element of $\mathcal{C}_A(X)$, while $\triangle_X$ is the least element of $\mathcal{C}_A(X)$.

4 Quotient of an action. For a left A-set X, we write $\mathrm{Quot}(X)$ for the class of all surjections of left A-sets with domain X. Recall that surjections $f_Y : X \to Y$ of left A-sets are pre-ordered by setting $f_Y \leq f_{Y'}$ if $f_Y = kf_{Y'}$ for some $k : Y' \to Y$; two such surjections are *equivalent* if $f_Y \leq f_{Y'}$ and $f_{Y'} \leq f_Y$ (i.e., if there exists a bijection $l : Y \to Y'$ of left A-sets with $f_Y = lf_{Y'}$), and the equivalence classes are called a *quotient* of the left A-set X. For any surjection $f_Y : X \to Y$ of left A-sets, the corresponding quotient will be denoted by $[(Y, f_Y)]$. We will say that a left A-set Y is a *quotient* of a left A-set X if there is a surjection $X \to Y$ of left A-sets. We let $\mathsf{Quot}(X)$ denote the class of all quotients of X.

Lemma 1. *For any monoid A, the passage from $(X, f_X) \in \mathsf{Quot}(\mathrm{A})$ to $\mathsf{K}[f_X]$ establishes a bijection $\mathsf{Quot}(\mathrm{A}) \simeq \mathcal{C}^l(A)$. Its inverse takes $\wp \in \mathcal{C}^l(A)$ to $(A/\wp, \pi^{\wp})$.*

5 Transversals of congruences. Let $\wp$ be a congruence on a left A-set X. A *transversal* of $\wp$ is a set $T \subseteq X$ such that T consists of exactly one representative of every equivalence class of $\wp$. In other words, $T \subseteq X$ is a transversal of $\wp$ if $\wp \cap (X \times X) = \triangle_Y$ and $\bigcup_{x \in X} [x]_{\wp} = X$. This is equivalent to saying that there exists a set-theoretical section of the canonical map $\pi^{\wp} : A \to A/\wp$ (i.e. a set-theoretical map $j : A/\wp \to A$ with

$\pi^{\wp} j = \mathsf{Id}_{A/\wp}$) such that its image in A is T. We write $\mathcal{C}_A(X)|_T$ for the subset of $\mathcal{C}_A(X)$ consisting of those congruences $\wp \in \mathcal{C}_A(X)$ for which T is transversal of $\wp$. Thus,

$$\mathcal{C}_A(X)|_T = \{\wp \in \mathcal{C}_A(X) : T \text{ is a transversal of } \wp\}.$$

6 Descent cohomologies. Given a morphism $\iota : B \to A$ of monoids, we write $\mathcal{Z}^l(\iota)$ for the set of maps $q : A \to B$ such that

(ZL1) $q\iota = \mathsf{Id}_B$;

(ZL2) $q(\iota(b)a) = bq(a)$, for all $b \in B$ and all $a \in A$, and

(ZL3) $q(aa') = q(a \cdot \iota q(a'))$, for all $a, a' \in A$.

The elements of set $\mathcal{Z}^l(\iota)$ are called *1-dimensional descent cocycles* (see [2]).

Lemma 2. *Let $\iota : B \to A$ be a homomorphism of monoids. Then for any $q \in \mathcal{Z}^l(\iota)$, $\mathsf{K}[q] \in \mathcal{C}^l(A)|_{\iota(B)}$.*

Given an injective homomorphism of monoids $\iota : B \to A$ and $\wp \in \mathcal{C}^l(A)|_{\iota(B)}$, write $q_\wp$ for the map $A \to B$ sending $a \in A$ to the unique element $b \in B$ with $[a]_\wp = [\iota(b)]_\wp$ (which exists because $\iota(B)$ is a transversal of $\wp$). Observe that $q_\wp \in \mathcal{Z}^l(\iota)$.

Based on the lemmas above, we have the following result.

Theorem 1. *Let $\iota : B \to A$ be an injective homomorphism of monoids. Then the map*

$$\mathcal{Z}^l(\iota) \to \mathcal{C}^l(A)|_{\iota(B)},$$

$$(q : A \to B) \longmapsto \mathsf{K}[q]$$

is a bijection. Its inverse is the map

$$\mathcal{C}^l(A)|_{\iota(B)} \to \mathcal{Z}^l(\iota),$$

$$\wp \longmapsto q_\wp.$$

By using Theorem 1 we will prove our main result:

Theorem 2. *For any monoid A, the assignment*

$$(\alpha, \beta) \longmapsto ([1_A]_\beta \xrightarrow{\iota_{[1_A]_\beta}} A \xleftarrow{\iota_{[1_A]_\alpha}} [1_A]_\alpha)$$

yields a bijection between the set of those pairs $(\alpha, \beta) \in \mathcal{C}^l(A) \times \mathcal{C}^r(A)$ such that

1. *$\alpha \cap \beta = \triangle_A$;*
2. *$[1_A]_\beta$ is a transversal of α,*
3. *$[1_A]_\alpha$ is a transversal of β*

and the set $\mathsf{FAC}(A)$ of factorizations of A.

R E F E R E N C E S

1. Kilp, M., Knauer, U., Mikhalev, A.V. Monoids, acts and categories. A handbook for students and researchers. *De Gruyter Expositions in Mathematics*, **29**. Walter de Gruyter & Co., Berlin, 2000.

2. Mesablishvili, B. On descent cohomology. *Transactions of A. Razmadze Mathematical Institute* **173** (2019), 137-155.

Received 20.05.2021; revised 30.06.2021; accepted 01.09.2021.

Author(s) address(es):

Tamar Mesablishvili
I. Javakhishvili Tbilisi State University
University str. 13, 0186 Tbilisi, Georgia
E-mail: tammoi14@gmail.com, tamar.mesablishvili392@ens.tsu.edu.ge