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ON NON-CLASSICAL SOLUTIONS FOR SOME NON-LOCAL BITSADZE-SAMARSKI BOUNDARY VALUE PROBLEM

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Abstract. In the paper, based on the variational approach, the definition of a classical solution is generalized for the simplest non-local boundary problem posed in a rectangular area.

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Let us consider the following non-local boundary value problem

$$\begin{gathered} -\triangle v(x,y)+\lambda v(x,y)=f(x,y), \quad (x,y)\in G, \\ v(x,y)|_{\Gamma}=0, \\ v(x,y)|_{\Gamma_{-\xi}}=v(x,y)|_{\Gamma_0}, \end{gathered} \tag{1}$$

where $G=\{(x,y)|-a<x<0, 0<y<b\}$ and ∂G is the boundary of the rectangle G, while Γ_t denotes the intersection of the line $x=t$ $(-a\leqslant t\leqslant 0)$ with $\overline{G}=G\cup\partial G$, $v(x,y)$ is the unknown and $f(x,y)$ is a given function, $\lambda=Const\geqslant 0, \Gamma=\partial G\setminus\Gamma_0$ and $\xi\in]0,a[$. Many scientific works are dedicated to the investigation of nonlocal problems (see, for example [2]-[6]). In some papers the possibility of variational formulations of such type problems for partial differential [4] and ordinary differential equations [5], [6] are studied.

It is known that if $f(x,y)\in C(\overline{G})$, then problem (1), has a unique classical solution $u(x,y)\in C^2(G)\cap C(\overline{G})$ [2],[3]. The aim of the paper is to generalize the definition of the classical solution to some of the discontinuous $f(x,y)$ functions on the right hand side.

Let us introduce some notations, definitions and facts [4]. Suppose $g(x,y)$ and $g_0(y)$ are defined almost everywhere on $G\setminus\Gamma_0$ and Γ_0 respectively. In addition, let $g(x,y)\in L_2(G)$ and $g_0(y)\in L_2(\Gamma_0)$. Construct the function $\overline{g}(x,y)$, defined almost everywhere on $\overline{G}$ as follows: restriction of $g(x,y)$ on $\overline{G}\setminus\Gamma_0$ is equivalent to $g(x,y)$, and restriction on Γ_0 (or the boundary value of $\overline{g}(0,y)$) to $g_0(y)$. Subsequently, each such $\overline{g}(x,y)$ function, defined almost everywhere on $\overline{G}$ will be identified with its generator $g(x,y)$ and $g_0(y)$, $\overline{g}(x,y)=(g(x,y),g_0(y))$. Denote the lineal of all such functions (pairs) by $D(\overline{G})$.

Suppose that $Q=\{(x,y)|0<x<\xi, 0<y<b\}$, and on $D(\overline{G})$ define a symmetric extension operator τ that for any function $\overline{v}(x,y)=(v(x,y),v_0(y))$ of the lineal $D(\overline{G})$ corresponds to the function in the rectangle $\overline{G}\cup\overline{Q}$

$$\tau\overline{v}(x,y)=\begin{cases} v(x,y), & if \quad (x,y)\in\overline{G}\setminus\Gamma_0, \\ -v(-x,y)+2v_0(y), & if \quad (x,y)\in\overline{Q} \end{cases}$$

equivalent to the function $\tau\overline{v}(x,y)$ defined almost everywhere. It is called the symmetric extension of the function $\overline{v}(x,y)$. Hereafter, we will use the notation $\tau\overline{v}(x,y)=\tilde{v}(x,y)$. Determine the scalar production on $D(\overline{G})$ lineal by $[\overline{g},\overline{h}]=\int_0^b\int_{-\xi}^{\xi}\int_{-a}^{x}\tilde{g}(s,y)\tilde{h}(s,y)dsdxdy$.

As a result, $D(\overline{G})$ lineal becomes the preimage of Hilbert space $H(\overline{G})$ with the norm $\|\overline{v}\|_H = [\overline{v},\overline{v}]^{\frac{1}{2}}$, that is equivalent to the norm $[]\overline{v}[]^2 = \|v(x,y)\|^2_{L_2(G)} + \|v_0(y)\|^2_{L_2(0,b)}$.

Thus $H(\overline{G})$ is the Hilbert space. To indicate that, for example, the function $\overline{g}(x,y)$ belongs to the space $H(\overline{G})$ we often use $\overline{g}(x,y) = (g(x,y), g_0(y)) \in H(\overline{G})$.

Suppose the domain of the operator $A = -\triangle + \lambda I$ is $D_A(\overline{G})$ lineal of the space $H(\overline{G})$, for which the following conditions are fulled for each function $\overline{v}(x,y)$:

1. $\overline{v}(x,y) \in C^2(\overline{G}), \quad \left.\dfrac{\partial^2 \overline{v}(x,y)}{\partial x^2}\right|_{\Gamma_{-\xi}} = 0, \quad \left.\dfrac{\partial^2 \overline{v}(x,y)}{\partial x^k}\right|_{\Gamma_0} = 0, \quad \forall y \in [0,b], \quad k = 1,2.$
2. $\overline{v}(x,y)|_{\Gamma} = 0, \quad \overline{v}(x,y)|_{\Gamma_{-\xi}} = \overline{v}(x,y)|_{\Gamma_0}.$

The lineal $D_A(\overline{G})$ is dense at $H(\overline{G})$ and the operator $A = -\triangle + \lambda I$ is positively defined on $D_A(\overline{G})$ lineal. Thus, we are able to follow the standard path of completion of $D_A(\overline{G})$ lineal into energy space [1]. Define the scalar product on $D_A(\overline{G})$ lineal by $[\overline{g},\overline{h}]_A = \int_0^b \int_{-\xi}^{\xi} \int_{-a}^{x} \left(\frac{\partial \tilde{g}(s,y)}{\partial s} \frac{\partial \tilde{h}(s,y)}{\partial s} + \frac{\partial \tilde{g}(s,y)}{\partial y} \frac{\partial \tilde{h}(s,y)}{\partial y} + \lambda \tilde{g}(s,y)\tilde{h}(s,y) \right) dsdxdy$ and for the corresponding norm use the notation $\|\cdot\|_A$. Introducing the scalar multiplication $D_A(\overline{G})$ lineal is transformed into the Hilbert space. Denote it by $S_A(\overline{G})$. Let us denote the energy space obtained by completing the space $S_A(\overline{G})$ by $H_A(\overline{G})$. In this space the norm defined by $|\|\overline{v}\||^2 = \|v\|^2_{W_2^{(1)}(G)} + \|v_0(y)\|^2_{W_2^{(1)}(0,b)}$ is equivalent to the norm $\|\cdot\|_A$.

Thus, for any function $\overline{v}(x,y) = (v(x,y), v_0(y)) \in H(\overline{G})$ of the space $H_A(\overline{G})$ we have $v(x,y) \in W_2^{(1)}(G)$, and $\overline{v}(0,y) = v_0(y)$ boundary value and the traces $v|_{\Gamma_{-\xi}}$ and $v|_{\Gamma_0}$ of the function $v(x,y)$ are equal and absolutely continuous ($v_0(y) \in W_2^{(1)}(0,b)$).

Let us $\varphi(y) \in L_2(0,b)$, $f(x,y) \in L_2(G)$, then for the function $\overline{f}(x,y) = (f(x,y), \varphi(y)) \in H(\overline{G})$ the quadratic functional $I_{f\varphi}(\overline{v}) = [\overline{v},\overline{v}]_A - 2[\overline{f},\overline{v}]$ has the unique minimizing function $\overline{u}_{f\varphi}(x,y) \in H_A(\overline{G})$ that for any $\overline{v}(x,y) \in H_A(\overline{G})$ satisfies $[\overline{u}_{f\varphi},\overline{v}]_A = [\overline{f},\overline{v}]$. Also there is constant $C_0 > 0$ such that $\|\overline{u}_{f\varphi}\|_A \leqslant C_0\|\overline{f}\|_H$. Based on this and on the equivalence of norms ($|\|\cdot\|| \sim \|\cdot\|_A, [] \cdot [] \backsim \|\cdot\|_H$) there exists a constant $C_1 > 0$ for which

$$\|\overline{u}_{f\varphi}\|^2_{W_2^{(1)}(G)} + \|\overline{u}_{f\varphi(0,y)}\|^2_{W_2^{(1)}(0,b)} \leq C_1 \left(\|f\|^2_{L_2(G)} + \|\varphi\|^2_{L_2(0,b)} \right). \tag{2}$$

It is worth noting that $\varphi(y)$ does not participate in the statement of the problem(1), it belongs to the space $L_2(0,b)$ (as a function parameter) and there is correspondence to each of its specific representatives (value) and to the unique minimizing function $\overline{u}_{f\varpi}(x,y) \in H_A(\overline{G})$ of the functional $I_{f\varphi}$. When $f(x,y) \in C(\overline{G})$ and $\varphi(y)$ are such that the corresponding minimizing function $\overline{u}_{f\varphi}(x,y)$ is smooth enough, then based on the equation $[\overline{u}_{f\varphi},\overline{v}]_A = [\overline{f},\overline{v}]$ it is obtained that $\overline{u}_{f\varphi}(x,y)$ is the classical solution of the problem (1): $\overline{u}_{f\varphi}(x,y) = u(x,y)$. Hereafter we will call $\varphi(y)$ the coordination function parameter. The following question arises. Is there $\varphi_f(y)$ value of the function parameter for which the minimizing function is the solution of the problem (1).

Theorem 1. *Suppose $\overline{f}(x,y) = (f(x,y), \varphi(y))$, where $f(x,y) \in C(\overline{G})$ and $\varphi(y) \in L_2(0,b)$ is the function parameter. In order to coincides the function $\overline{u}_{f\varphi}(x,y) \in H_A(\overline{G})$ with the* *solution of problem (1), it is necessary and sufficient that*

$$-\frac{d^2\overline{u}_{f\varphi}(0,y)}{dy^2}+\lambda\overline{u}_{f\varphi}(0,y)=\varphi(y),\quad y\in]0,b[.\tag{3}$$

Based on Theorem 1, when $f(x,y)\in C(\overline{G})$ and $\varphi(y)=-\frac{d^2u(0,y)}{dy^2}+\lambda u(0,y)$ the minimizing function $\overline{u}_{f\varphi}(x,y)$ of $I_{f\varphi}$ will coincide with the classical solution $u(x,y)$ of problem (1). In particular, if $f(x,y)\in R_A(\overline{G})$, we have $\varphi(y)=f(-\xi,y)$.

Theorem 1 is fair even when we change equation (3) to its integral form.

Theorem 2. *Suppose* $\overline{f}(x,y)=(f(x,y),\varphi(y)),f(x,y)\in C(\overline{G}),\varphi(y)\in L_2(0,b)$ *is the value of the function parameter. In order the minimizing function* $\overline{u}_{f\varphi}(x,y)\in H_A(\overline{G})$ *coincides with the solution of problem* (1)*, it is necessary and sufficient to have*

$$\int_0^b\left(\frac{d\overline{u}_{f\varphi}(0,y)}{dy}\frac{d\eta(y)}{dy}+\lambda\overline{u}_{f\varphi}(0,y)\eta(y)\right)dy=\int_0^b\varphi(y)\eta(y)dy,\forall\eta(y)\in\overset{0}{W}{}_2^{(1)}(0,b).\tag{4}$$

According to Theorem 2, if $f(x,y)\in C(\overline{G})$, then the following two problems are equivalent:

I. Find the classical solution of problem (1);

II. Find the minimizing function $\overline{u}_{f\varphi}(x,y)\in H_A(\overline{G})$ of the functional $J_{f\varphi}(\overline{v})$, for which the trace $\overline{u}_{f\varphi}(0,y)\in\overset{0}{W}{}_2^{(1)}(0,b)$ and the value of the parameter function $\varphi(y)\in L_2(0,b)$ satisfy equality (4).

It should be noted that the second problem can be posed in a more general way if we replace $f(x,y)\in C(\overline{G})$ by $f(x,y)\in L_2(G)$. Thus, in order to generalize the concept of the classical solution of problem (1) it is desirable to introduce the following definition.

Definition. Suppose that for $f(x,y)\in L_2(G)$, there exists a function $\varphi_f(y)\in L_2(0,b)$ such that $\overline{f}(x,y)=(f(x,y),\varphi_f(y))\in H_A(\overline{G})$ and the trace $\overline{u}_{f\varphi_f}(0,y)\in\overset{0}{W}{}_2^{(1)}(0,b)$ of the function $\overline{u}_{f\varphi_f}(x,y)$ minimizing the functional $J_{f\varphi_f}(\overline{v})$ in the space $H_A(\overline{G})$ satisfies (4), then such $\overline{u}_{f\varphi_f}(x,y)\in H_A(\overline{G})$ is called a generalized solution of problem (1).

Hereinafter, $\varphi_f(y)$ is called the function of the right-hand side $f(x,y)$ and $\overline{f}(x,y)=(f(x,y),\varphi_f(y))\in H(\overline{G})$ the coordinated pair of (1).

It is easy to see that, if there exists a generalized solution of problem (1), then it is unique. Hereinafter, it is denoted by $\overline{u}(x,y)$. It is clear that if $f(x,y)\in C(\overline{G})$ then the generalized solution of problem (1) coincides with the classical solution: $\overline{u}(x,y)=u(x,y)$.

According to the inequality $\|\overline{u}_{f\varphi}\|_A\leqslant C_0\|\overline{f}\|_H$, there exists a constant $C_0>0$ such that

$$\|\overline{u}(x,y)\|_A\leq C_0\|(f(x,y),\varphi_f(y))\|_H.\tag{5}$$

(5) expresses continuous dependence of the generalized solution on the coordinated pair.

Theorem 3. *Suppose* $\overline{f}_n(x,y)=(f_n(x,y),\varphi_n(y))\in H(\overline{G}),n=1,2...$ *is a fundamental sequence of coordinated pairs, and* $\lim_{n\to\propto}\|f_n(x,y)-f(x,y)\|_{L_2(G)}=0$*. Besides, let* $\{\overline{u}_n(x,y)\}$ *be the sequence of the proper generalized solutions of these coordinate pairs. Then* $\left\{\overline{f}_n(x,y)\right\}$ *converges to the coordinated pair of problem* (1) *and the sequence of the generalized solutions* $\{u_n(x,y)\}$ *converges to the general solution of problem* (1).

To illustrate that the above definition of a generalized solution of problem (1) generalizes the concept of a classical solution, let us give a example of the nonclassical solution of problem (1). Take a function $\overline{u}(x,y) \in H_A(\overline{G})$ that does not have the smoothness of a classical solution (for example, a second-order derivative has a discontinuity on some curve). With $\overline{u}(x,y)$ we are finding "corresponding" $\overline{f}(x,y) = (f(x,y), \varphi_f(y)) \in H_A(\overline{G})$ pairs: $f(x,y) = -\triangle \overline{u}(x,y) + \lambda \overline{u}(x,y)$ almost everywhere on the G and at the same time relation (4) is satisfied (at the expense of choosing proper $\overline{u}(x,y)$). In the simple cases $\overline{u}(0,y)$ will be smooth and $\varphi_f(y) = -\frac{d^2\overline{u}(0,y)}{dy^2} + \lambda \overline{u}(0,y)$. It is likely that $\overline{u}(x,y)$, chosen in this way, will be a generalized solution of problem (1). To prove this, it is sufficient to show that the variation $\delta J_{f\varphi_f}(\overline{u}, \overline{h}) = 0$.

In the example below, we took $\lambda = 0, \xi = \frac{1}{2}$ and $a = b = 1$.

Example. Suppose $\overline{G} = \overline{G_1} \cup \overline{G_2}$. Consider

$$\overline{u}(x,y) = \begin{cases} 4x(x+1)y(y-1), & when \quad (x,y) \in \overline{G}_1, \\ -y^2 + y, & when \quad (x,y) \in \overline{G} \setminus \overline{G}_1. \end{cases} \tag{6}$$

Accordingly, $f(x,y) = \begin{cases} -8(x^2 + x + y^2 + y), & when \quad (x,y) \in \overline{G}_1, \\ 2, & when \quad (x,y) \in \overline{G} \setminus \overline{G}_1 \end{cases}$

and $\varphi_f(y) = 2, y \in [0,1]$.

It is obvious that condition (4) is fulfilled. The fact that the corresponding variation is zero can be verified easily. Thus, the function (6) is indeed a generalized solution ($f(x,y)$, is not continuous on $\Gamma_{-\frac{1}{2}}$ and obviously $\overline{u}(x,y)$ cannot be a classical solution).

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