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ONE EXAMPLE OF m - DEPENDENT VECTOR'S SEQUENCE

Zurab Kvatadze Beqnu Parjiani Tsiala Kvatadze

Abstract. By means of the functions defined on the interval $[0,1[$ a regular finite Markov chain is constructed. On the square $[0,1[\times[0,1[$ a sequence $\{T_n\}_{n\geq 1}$ of conditionally m-dependent vectors controlled by this chain is considered. The limiting distribution of the sequence of sums $S_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[T_i - ET_i]$, $n = 1, 2, \ldots$ is determined.

Keywords and phrases: Markov's chain, conditionally m-Dependent vectors, sequence with chain dependence.

1 Introduction.

Let us consider a stationary in the narrow sense two-component sequence

$$\{\xi_i, Y_i\}_{i\geq 1} \tag{1}$$

defined on a probabilistic space (Ω, F, P), where $\xi_i : \Omega \to \Xi$, and $Y_i : \Omega \to R^k$.

Definition. The sequence $\{Y_i\}_{i\geq 1}$ from (1) is called as conditionally m-dependent sequence, ([1]) if the vectors $Y_1, Y_2, ..., Y_n$ on a fixed trajectory $\bar{\xi}_{1n} = (\xi_1, \xi_2, .., \xi_n)$ for any natural number n become independent when the difference of their indices exceeds m. Moreover, the distribution of Y_i depends only on ξ_i .I.e. for any natural numbers i, l, n, $j_1, j_2, \ldots, j_l$, $(2 \leq l \leq n.\quad i \leq n;\ 1 \leq j_1 < j_2 < ... < j_l \leq n)$ the following equalities are valid:

$$P_{(Y_j, Y_{j_2}, ..., Y_{j_r})|\bar{\xi}_{1n}} = P_{Y_{j_1}|\xi_{j_1}} * P_{Y_{j_2}|\xi_{j_2}} * ... * P_{Y_{j_r}|\xi_{j_r}}, \quad if \quad |j_k - j_l| > m.$$

$$P_{Y_i|\bar{\xi}_{1n}} = P_{Y_i|\xi_i}, \qquad 1 < j_1 < j_2 < ... < j_l < n, \quad i = \overline{1, n}$$

If $\{\xi_i\}_{i\geq 1}$ is a Markov chain with discrete time, then the sequence $\{Y_i\}_{i\geq 1}$ is a conditionally m-dependent sequence with a control Markov chain ([1]).
when m=0, a conditionally m dependent sequence represents a conditionally independent sequence (see [2]).

Suppose $\{\xi_i\}_{i\geq 1}$ for each function $\Psi : \Xi \to R^1$ for which $E\Psi(\xi_1) < \infty$, when $n \to \infty$ with probability 1, there is a convergence of

$$\frac{1}{n}\sum_{j=1}^{n}\Psi(\xi_j) \to E\Psi(\xi_1) \tag{2}$$

2 Central limit theorem. Let's introduce the values:

$$\mu(\xi_j) = E(Y_j|\xi_j), \qquad \mu = E\mu(\xi_1) = EY_1,$$
$$R(\xi_j,\xi_l) = E\{[Y_j - \mu(\xi_j)][Y_l - \mu(\xi_l)]^T|\xi_{1n}\}, \qquad 1 \le j,l \le n$$
$$R_0^{(l)} = ER(\xi_1,\xi_{1+l}), \qquad R_0^{(-l)} = ER(\xi_{1+l},\xi_1), \qquad l = 0,...,m$$
$$R_m = \sum_{l=-m}^{m} R_0^{(l)} = R_0^{(0)} + \sum_{l=1}^{m}[R_0^{(l)} + (R_0^{(l)})^T], \tag{3}$$

Let's consider the expansion $S_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}[Y_j - \mu] = S_{n_1} + S_{n_2}$

$$S_{n_1} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}[Y_j - \mu(\xi_j)] \qquad S_{n_2} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}[\mu(\xi_j) - \mu]$$

Theorem 1. *(see [1]) When $\{Y_i\}_{i\ge1}$ is a conditionally m dependent sequence from (1) satisfies a condition (2) and $sp(R_m) < \infty$, then:*

a) $P_{S_{n_1}|\xi_{1n}.} \xrightarrow{W} \Phi_{R_m}$ a.e. b) $P_{S_{n_1}} \xrightarrow{W} \Phi_{R_m}$,

c) If $P_{S_{n_2}} \xrightarrow{W} Q$, then $P_{S_n} \xrightarrow{W} \Phi_{R_m} * Q$.

Suppose $\{\xi_i\}_{i\ge1}$ is a stationary, homogeneous, finite, ergodic Markov chain with a single ergodic class (possibly in cyclic subclasses). Suppose there is a set of states $\{b_1, b_2, ..., b_r\}$, a transition matrix is $P = \|P_{\alpha\beta}\|_{\alpha\beta=1,\overline{r}}$. The vector of limiting stationary distribution is $\pi = (\pi_1, \pi_2, \ldots, \pi_r)$.

A fundamental matrix is $Z = I + (\sum_{j=1}^{\infty}(P^j - \Pi))_c = \|z_{\alpha\beta}\|_{\alpha,\beta=\overline{1,r}}$, where

$$\Pi = \begin{pmatrix} \pi_1, \pi_2, ..., \pi_r \\ .... \\ \pi_1, \pi_2, ..., \pi_r \end{pmatrix} = \|\pi_{\alpha\beta}\|_{\alpha,\beta=1,\overline{r}}, \qquad \pi_{\alpha\beta} = \pi_\beta, \qquad \alpha,\beta = \overline{1,r}.$$

Let's introduce the designations:

$$\mu(\alpha) = E(Y_1|\xi_1 = b_\alpha) = (\mu_1(\alpha), \mu_2(\alpha), ..., \mu_k(\alpha)), \qquad \alpha = \overline{1,r}.$$
$$T_\mu = \|t_{\mu_{i,j}}\|_{i,j=\overline{1,k}}, t_{\mu_{i,j}} = \sum_{\alpha,\beta=1}^{r}(\pi_\alpha z_{\alpha\beta} + \pi_\beta z_{\beta\alpha} - \pi_\alpha\pi_\beta - \pi_\alpha\delta_{\alpha\beta})\mu_i(\alpha)\mu_j(\beta), \qquad i,j = \overline{1,k}.$$
$$F = \|\mu_{ij}\|_{\substack{i=1,k \\ j=1,r}} \qquad \mu_{ij} = \mu_i(b_j), i = \overline{1,k}, \qquad j = \overline{1,r}.$$

When $k = 1$ and the chain is a regular image of T_μ is determined in [4]. In [3] the representation of T_μ as a matrix is obtained, when $k > 1$ and the chain is ergodic with one ergodicity class that would contain cyclic subclasses

$$T_\mu = F[\Pi_{dg}Z + (\Pi_{dg}Z)^T - \Pi_{dg}\Pi - \Pi_{dg}]F^T. \tag{4}$$

Using this fact, the result obtained for a regular chain [1] would be generalized to a single ergodic class for an ergodic chain.

Theorem 2. *When by the conditions of Theorem 1 $\{\xi_i\}_{i\geq 1}$ is a homogeneous stationary ergodic chain with one class of ergodicity and above stated characteristics, then with the points a) and b) of the Theorem, the following is valid:*

$$c)\mathrm{P}_{S_{n_2}} \xrightarrow{W} \Phi_{T_\mu}, \quad d)P_{S_n} \xrightarrow{W} \Phi_{R_m+T_\mu}.$$

Example. On the interval $[0,1[$ let's consider the sequence of functions $\{\tau_n(\omega)\}_{n\geq 1}$

$$\tau_n(\omega) = 0 \cdot I_{(\omega)\in\bigcup_{i=1}^{2^{n-1}}[\frac{2(i-1)}{2^n}, \quad \frac{2i-1}{2^n}]} + 1 \cdot I_{\omega\in\bigcup_{i=1}^{2^{n-1}}[\frac{2i-1}{2^n}, \quad \frac{2i}{2^n}].)}$$

$\{\xi_n\}_{n\geq 1}, \quad \xi_n = \tau_n + \tau_{n+1}$ is Markov's finite, stationary chain. The set of states is $\{0,1,2\}$. The initial and limiting distribution vector is $\pi = (1/4,\ 1/2,\ 1/4)$. P and Z are the probability transition matrix and the fundamental matrix

$$P = \begin{pmatrix} 12 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}, \quad Z = \begin{pmatrix} 3/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 3/2 \end{pmatrix}.$$

Let's define the following $\{X_n\}_{n\geq 1}$, $\{Y_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$ sequences on the square $[0,1[\times[0,1[$:

$$X_n(\omega_1,\omega_2) = \{(\alpha,\beta),\ \alpha,\beta \in \overline{1,2^{m+1}},\ if \mid \begin{matrix} \omega_1 \in \bigcup_{j=1}^{30^{n-1}}[\frac{j-1}{30^{n-1}} + \frac{\alpha-1}{30^{n-1}.2^{m+1}}, \frac{j-1}{30^{n-1}} + \frac{\alpha}{30^{n-1}.2^{m+1}}[\\ \omega_2 \in \bigcup_{j=1}^{30^{n-1}}[\frac{j-1}{30^{n-1}} + \frac{\beta-1}{30^{n-1}.2^{m+1}}, \frac{j-1}{30^{n-1}} + \frac{\beta}{30^{n-1}.2^{m+1}}[\end{matrix}.$$

$$Y_n(\omega_1,\omega_2) = \{(\alpha,\beta),\ \alpha,\beta \in \overline{1,3^{m+1}},\ if \mid \begin{matrix} \omega_1 \in \bigcup_{j=1}^{30^{n-1}}[\frac{j-1}{30^{n-1}} + \frac{\alpha-1}{30^{n-1}.3^{m+1}}, \frac{j-1}{30^{n-1}} + \frac{\alpha}{30^{n-1}.3^{m+1}}[\\ \omega_2 \in \bigcup_{j=1}^{30^{n-1}}[\frac{j-1}{30^{n-1}} + \frac{\beta-1}{30^{n-1}.3^{m+1}}, \frac{j-1}{30^{n-1}} + \frac{\beta}{30^{n-1}.3^{m+1}}[\end{matrix}.$$

$$Z_n(\omega_1,\omega_2) = \{(\alpha,\beta),\ \alpha,\beta \in \overline{1,5^{m+1}},\ if \mid \begin{matrix} \omega_1 \in \bigcup_{j=1}^{30^{n-1}}[\frac{j-1}{30^{n-1}} + \frac{\alpha-1}{30^{n-1}.5^{m+1}}, \frac{j-1}{30^{n-1}} + \frac{\alpha}{30^{n-1}.5^{m+1}}[\\ \omega_2 \in \bigcup_{j=1}^{30^{n-1}}[\frac{j-1}{30^{n-1}} + \frac{\beta-1}{30^{n-1}.5^{m+1}}, \frac{j-1}{30^{n-1}} + \frac{\beta}{30^{n-1}.5^{m+1}}[\end{matrix}.$$

Each of them is a sequence of uniformly distributed vectors

$$P\{X_n = (\alpha,\beta\} = 1/2^{2(m+1)}, \quad if \quad (\alpha,\beta) \in A, \quad P\{Y_n = (\alpha,\beta)\} = 1/3^{2(m+1)}, \quad (\alpha,\beta) \in B,$$

$$P\{Z_n = (\alpha,\beta)\} = 1/5^{2(m+1)}, \quad if \quad (\alpha,\beta) \in C$$

where

$$A = \{(\alpha,\beta)|1 \leq \alpha,\beta \leq 2^{m+1}\}, B = \{(\alpha,\beta)|1 \leq \alpha,\beta \leq 3^{m+1}\}, C = \{(\alpha,\beta)|1 \leq \alpha,\beta \leq 5^{m+1}\}.$$

Sequence $\{T_n\}_{n\geq 1}$, $T_n = X_n I_{(\xi_n=0)} + Y_n I_{(\xi_n=1)} + Z_n I_{(\xi_n=2)}$ is a conditionally m-dependent sequence of uniformly distributed vectors.

Let's consider the case $m = 1$. According to Theorem 2, it is possible to determine the limiting distribution of the sum $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [T_i - ET_1]$.

$$P_{S_n} \xrightarrow{W} \Phi_{R_1+T_\mu}.$$

From (2.1) and (2.2) equalities R_1 and T_μ matrices we have

$$R_1 + T_\mu = \begin{pmatrix} 183283/2400 & 211441/4800 \\ 211441/4800 & 183283/2400 \end{pmatrix}$$

The following sequences are possible: $\{X_n\}_{n\geq 1}$, $\{Y_n\}_{n\geq 1}$, $\{Z_n\}_{n\geq 1}$ and $\{T_n\}_{n\geq 1}$ to be applied in geological surveys. Due to them obtained from pits data will be defined contamination of minerals in survey areas.

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Author(s) address(es):

Zurab Kvatadze, Beqnu Parjiani
Department of Mathematics, Georgian Technical University
Kostava str. 77, 0171 Tbilisi, Georgia
E-mail: zurakvatadze@yahoo.com, beqnufarjiani@yahoo.com

Tsiala Kvatadze
Department of Mathematics, Georgian Primary Program, European School
Skhirtladze str. 2, 0177, Tbilisi, Georgia
E-mail: ttkvatadze@gmail.com