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UTILITY MAXIMIZATION PROBLEM UNDER BINOMIAL MODEL UNCERTAINTY

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Abstract. We provide the exact solution of the robust portfolio optimization problem under model uncertainty with power utility.

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Let r be interest rate of riskless asset and let $p_0, p_1, 0 < p_0 < p_1 < 1$ be possible probability of growth of a risky asset. Let $S_n, n = 1, 2, ..., N$ be the risky asset price defined by

$$S_{n+1}(\omega_1...\omega_n\omega_{n+1}) = (1_{(\omega_{n+1}=H)}u + 1_{(\omega_{n+1}=T)}d)S_n(\omega_1...\omega_n),$$

where $\omega = \omega_1...\omega_N \in \{H, T\}^N,\ \tilde{q} = \frac{u-1-r}{u-d},\ \tilde{p} = \frac{1+r-d}{u-d}$.

Let $\tilde{P}(\omega) = \tilde{p}^{\#H(\omega)}(1-\tilde{p})^{\#T(\omega)}$ be risk-neutral probability measure on $\{H, T\}^N$ and let $P_i(\omega) = p_i^{\#H(\omega)}(1-p_i)^{\#T(\omega)}, i = 0, 1$ be reference measures, where $\#H(\omega), \#T(\omega)$ denotes number of Heads and Tails respectively. For density we have

$$Z_i = \frac{P_i}{\tilde{P}} = \left(\frac{p_i}{\tilde{p}}\right)^{\#H(\omega)} \left(\frac{1-p_i}{1-\tilde{p}}\right)^{\#T(\omega)} = \left(\frac{p_i}{\tilde{p}}\right)^{\#H(\omega)} \left(\frac{q_i}{\tilde{q}}\right)^{\#T(\omega)},\ i = 0, 1.$$

We use also the notation E_α for the expectation w.r.t. $Z_\alpha = \alpha Z_1 + (1-\alpha)Z_0, \alpha \in [0, 1]$.

We consider the robust utility maximization problem

$$\max_{\Delta} E_0U(X_N) \wedge E_1U(X_N),$$

for wealth process $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$, which is equivalent to

$$\max_{V:\tilde{E}V=X_0'} E_0U(V) \wedge E_1U(V),\ X_0' = X_0(1+r)^N.$$

It is Clear that $E_0U(V) \wedge E_1U(V) = \min_{\alpha\in[0,1]} E_\alpha U(V)$. Thus we get

$$\max_{V:\tilde{E}V=X_0'} E_0U(V) \wedge E_1U(V) = \max_{V:\tilde{E}V=X_0'} \min_{\alpha\in[0,1]} E_\alpha U(V) = \min_{\alpha\in[0,1]} \max_{V:\tilde{E}V=X_0'} E_\alpha U(V).$$

The existence of a solution of such problem is well known [1]. When $U(x)$ coincides with the power utility function $x^\gamma,\ \gamma < 1,\ \gamma \neq 0$, the solution of the utility maximization problem

$$\max_{V:\tilde{E}V=X_0'} E_\alpha U(V)$$

is equal to $V = X'_0 \frac{Z_\alpha^{\frac{1}{1-\gamma}}}{\tilde{E}Z_\alpha^{\frac{1}{1-\gamma}}}$ [2]. Hence

$$\max_{V=X'_0} E_\alpha U(V) = X_0'^\gamma \frac{1}{\tilde{E}^\gamma Z_\alpha^{\frac{1}{1-\gamma}}} \tilde{E}Z_\alpha (Z_\alpha^{\frac{1}{1-\gamma}})^\gamma = X_0'^\gamma \tilde{E}^{1-\gamma} Z_\alpha^{\frac{1}{1-\gamma}}$$

and it remains to solve the minimization problem

$$\min_{0\le\alpha\le1} \tilde{E}Z_\alpha^{\frac{1}{1-\gamma}},\ \gamma \neq 0.$$

Let $g(\alpha) = \tilde{E}(\alpha Z_1 + (1-\alpha)Z_0)^\beta$, where $\beta = \frac{1}{1-\gamma} \ge 1$ and let α^* be such that $\min_{0\le\alpha\le1} g(\alpha) = g(\alpha^*)$. It is clear that

$$\max_{V:\tilde{E}V=X'_0} E_0U(V) \wedge E_1U(V) = \max_{V:\tilde{E}V=X'_0} E_{\alpha^*}U(V) = X_0'^\gamma \tilde{E}^{1-\gamma} Z_{\alpha^*}^{\frac{1}{1-\gamma}}. \tag{1}$$

Theorem. *Let $\beta > 1$. Then the solution of maximization problem (1) is*

$$V = \frac{X'_0}{\tilde{E}Z_{\alpha^*}^{\frac{1}{1-\gamma}}} (\alpha^* Z_1 + (1-\alpha^*)Z_0)^{\frac{1}{1-\gamma}},\ \alpha^* = \begin{cases} 1,\ if\ \tilde{p} \le p_0 < p_1, \\ 0,\ if\ p_0 < p_1 \le \tilde{p}, \\ g'^{-1}(0),\ if\ \ p_0 < \tilde{p} < p_1. \end{cases}$$

Proof. Since $g''(\alpha) = \beta(\beta-1)\tilde{E}(\alpha Z_1 + (1-\alpha)Z_0)^{\beta-2}(Z_1 - Z_0)^2 > 0$, g is strictly convex. Hence α^* is unique and

$$\alpha^* = \begin{cases} 0,\ if\ g'(0) \ge 0, g'(1) \ge 0, \\ 1,\ if\ g'(0) \le 0, g'(1) \le 0, \\ g'^{-1}(0)\ if\ g'(0) < 0, g'(1) > 0. \end{cases}$$

We have $g'(\alpha) = \beta\tilde{E}(\alpha Z_1 + (1-\alpha)Z_0)^{\beta-1}(Z_1 - Z_0)$. Hence $g'(0) = \beta\tilde{E}Z_0^{\beta-1}(Z_1 - Z_0)$, $g'(1) = \beta\tilde{E}Z_1^{\beta-1}(Z_1 - Z_0)$ and

$$\alpha^* = \begin{cases} 0,\ if\ \tilde{E}Z_0^{\beta-1}(Z_1 - Z_0) \ge 0, \tilde{E}Z_1^{\beta-1}(Z_1 - Z_0) \ge 0, \\ 1,\ if\ \tilde{E}Z_0^{\beta-1}(Z_1 - Z_0) \le 0, \tilde{E}Z_1^{\beta-1}(Z_1 - Z_0) \le 0, \\ g'^{-1}(0),\ if\ \tilde{E}Z_0^\beta > \tilde{E}Z_0^{\beta-1}Z_1, \tilde{E}Z_1^\beta > \tilde{E}Z_1^{\beta-1}Z_0. \end{cases}$$

Since

$$\begin{aligned}
\tilde{E}Z_1^{\beta-1}Z_0 &= E_0 Z_1^{\beta-1} \\
&= \sum_{\omega} p_0^{\sharp H(\omega)}(1-p_0)^{\sharp T(\omega)} \left(\frac{p_1}{\tilde{p}}\right)^{\sharp H(\omega)(\beta-1)} \left(\frac{1-p_1}{1-\tilde{p}}\right)^{\sharp T(\omega)(\beta-1)} \\
&= \sum_{k=0}^{N} C_n^k \left(p_0 \left(\frac{p_1}{\tilde{p}}\right)^{\beta-1}\right)^k \left((1-p_0)\left(\frac{1-p_1}{1-\tilde{p}}\right)^{\beta-1}\right)^{N-k} \\
&= \left(\left(\frac{p_1}{\tilde{p}}\right)^{\beta-1} p_0 + \left(\frac{1-p_1}{1-\tilde{p}}\right)^{\beta-1}(1-p_0)\right)^N, \\
\tilde{E}Z_0^{\beta-1}Z_1 &= \left(\left(\frac{p_0}{\tilde{p}}\right)^{\beta-1} p_1 + \left(\frac{1-p_0}{1-\tilde{p}}\right)^{\beta-1}(1-p_1)\right)^N, \\
\tilde{E}Z_0^{\beta} &= \left(\left(\frac{p_0}{\tilde{p}}\right)^{\beta-1} p_0 + \left(\frac{1-p_0}{1-\tilde{p}}\right)^{\beta-1}(1-p_0)\right)^N, \\
\tilde{E}Z_1^{\beta} &= \left(\left(\frac{p_1}{\tilde{p}}\right)^{\beta-1} p_1 + \left(\frac{1-p_1}{1-\tilde{p}}\right)^{\beta-1}(1-p_1)\right)^N,
\end{aligned}$$

inequalities $g'(0) < 0,\ g'(1) > 0$ are equivalent to $\tilde{E}Z_0^{\beta-1}Z_1 < \tilde{E}Z_0^{\beta},\ \tilde{E}Z_1^{\beta} > \tilde{E}Z_1^{\beta-1}Z_0$ and

$$\begin{aligned}
\left(\frac{p_0}{\tilde{p}}\right)^{\beta-1} p_0 + \left(\frac{1-p_0}{1-\tilde{p}}\right)^{\beta-1}(1-p_0) &> \left(\frac{p_0}{\tilde{p}}\right)^{\beta-1} p_1 + \left(\frac{1-p_0}{1-\tilde{p}}\right)^{\beta-1}\Big(1-p_1\Big), \\
\left(\frac{p_1}{\tilde{p}}\right)^{\beta-1} p_1 + \left(\frac{1-p_1}{1-\tilde{p}}\right)^{\beta-1}(1-p_1) &> \left(\frac{p_1}{\tilde{p}}\right)^{\beta-1} p_0 + \left(\frac{1-p_1}{1-\tilde{p}}\right)^{\beta-1}(1-p_0).
\end{aligned}$$

Hence α^* belongs to $(0,1)$ if and only if

$$\begin{aligned}
\left(\frac{p_0}{\tilde{p}}\right)^{\beta-1}\Big(p_0-p_1\Big) &> \left(\frac{1-p_0}{1-\tilde{p}}\right)^{\beta-1}(p_0-p_1), \\
\left(\frac{p_1}{\tilde{p}}\right)^{\beta-1}(p_1-p_0) &> \left(\frac{1-p_1}{1-\tilde{p}}\right)^{\beta-1}(p_1-p_0)
\end{aligned}$$

or equivalently

$$\begin{aligned}
\left(\frac{p_0}{\tilde{p}}\right)^{\beta-1} &< \left(\frac{1-p_0}{1-\tilde{p}}\right)^{\beta-1}, \\
\left(\frac{p_1}{\tilde{p}}\right)^{\beta-1} &> \left(\frac{1-p_1}{1-\tilde{p}}\right)^{\beta-1}.
\end{aligned}$$

This it takes place if and only if $p_0 < \tilde{p} < p_1$. □

Example 1. For $\tilde{p} = \frac{1}{2},\ \gamma = \frac{1}{2}$ we have

$$\alpha^* = \begin{cases} 0,\ if\ \tilde{p} \le p_0 < p_1 \\ 1,\ if\ p_0 < p_1 \le \tilde{p}, \\ \dfrac{(p_1^2 + q_1^2)^N - (p_0 p_1 + q_0 q_1)^N}{(p_0^2 + q_0^2)^N - 2(p_0 p_1 + q_0 q_1)^N + (p_1^2 + q_1^2)^N},\ if\ p_0 < \tilde{p} < p_1, \end{cases}$$

since the solution of $\tilde{E}(\alpha Z_1 + (1-\alpha) Z_0)(Z_1 - Z_0) = 0$ is $\alpha^* = \frac{\tilde{E}(Z_0^2 - Z_1 Z_0)}{\tilde{E}(Z_1^2 - 2Z_1 Z_0 + Z_0^2)}$.

Example 2. If $\beta = 3$ (i.e. $\gamma = \frac{2}{3}$), then α^* is a root of equation $\tilde{E}(\alpha Z_1 + (1-\alpha)Z_0)^2(Z_1 - Z_0) = 0$ or equation

$$\tilde{E} Z_0^2 (Z_1 - Z_0) + 2\alpha \tilde{E} Z_0 (Z_1 - Z_0)^2 + \alpha^2 \tilde{E}(Z_1 - Z_0)^3 = 0.$$

i.e. $\alpha^* = \frac{-\tilde{E} Z_0 (Z_1 - Z_0)^2 + \sqrt{\tilde{E}^2 Z_0 (Z_1 - Z_0)^2 - \tilde{E}(Z_1 - Z_0)^3 \tilde{E} Z_0^2 (Z_1 - Z_0)}}{\tilde{E}(Z_1 - Z_0)^3}$.

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