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ON THE EXACT SOLUTION OF THE ZAKHAROV-KUZNETSOV TYPE NONLINEAR PARTIAL DIFFERENTIAL EQUATION

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Abstract. Using the exp-function method the traveling wave special exact solution of the (2+1)D nonlinear Zakharov-Kuznetsov type partial differential equation is obtained. It is shown that such a solution can be expressed through the hyperbolic tangent function and has spatially isolated structural (soliton-like) forms. Revision of previously obtained solutions is discussed.

Keywords and phrases: Traveling waves, exp-function method, exact solution, soliton.

AMS subject classification (2010): 76B25, 76D17, 35Q51, 74J40, 76L05.

1 Introduction. The Zakharov-Kuznetsov (ZK) equation governs the behavior of weakly nonlinear ion-acoustic waves in a magnetized plasma [1]. In the given paper, we consider the ZK-modified equal-width (ZK-MEW) equation. Using the straightforward method, soliton-like exact solutions are obtained by Tsamalashvili [2] for the 2D nonlinear modified Burger equation. Using the special exp-function method traveling wave exact solutions of the 2D nonlinear Burger equation are obtained by Tsamalashvili [3], showing the existence of spatially isolated structural (soliton-like) forms. Employing the special exp-function expansion method Mohyud-Din et al. [4] constructed exact traveling wave solutions for the ZK-MEW equation and the (2+1)D Burger equation. Unfortunately, this work contains numerous wrong results and our main purpose is to revise such solutions.

2 Solution of (2+1)-dimensional Zakharov-Kuznetsov-modified equal-width equation. Consider the nonlinear ZK-MEW equation

$$u_t + \alpha(u^2)_x + (\beta u_{xt} + \delta u_{yy})_x = 0, \tag{1}$$

where α, β and δ are some nonzero parameters. To find the traveling wave solutions we use $u = u(\eta)$, $\eta = x + y - Vt$ then we can convert Eq.(1) into the following ODE

$$-Vu' - \beta V u''' + \delta u''' + 2\alpha u u' = 0, \tag{2}$$

where the prime denotes the derivatives with respect to η. Now integrating Eq. (2), we have

$$-Vu + (\delta - \beta V)u'' + \alpha u^2 + c = 0. \tag{3}$$

The use of special exp-function expansion method [3] allows to find exact solutions of a nonlinear evolutionary equation (3) by the series of function $\exp(-n\varphi(\eta))$, where the $\varphi(\eta)$ function satisfies the ODE:

$$\frac{\partial \varphi(\eta)}{\partial \eta} = e^{-\varphi(\eta)} + \mu e^{\varphi(\eta)} + \lambda, \tag{4}$$

where, μ and λ are parameters. Thus, we are seeking the solution of Eq.(3) by the following finite series

$$u(\eta) = \sum_{n=0}^{M} a_n \exp(-n(\varphi(\eta)), \tag{5}$$

where a_n and $0 \leq n \leq M$ are constants and M is a homogenous balance number. It is clear that the solution of Eq.(3) depends on the relations between μ and λ. Namely, we will consider the following special case when $\lambda^2 - 4\mu > 0, \mu \neq 0$, when the solution of Eq.(4) can be given as

$$\varphi(\eta) = \ln\left\{\frac{1}{2\mu}\left[-\sqrt{\lambda^2 - 4\mu}\tanh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\eta + c)) - \lambda\right]\right\}. \tag{6}$$

The positive integer M can be determined by considering the homogeneous balance between the highest order derivatives u'' and nonlinear terms u^2 appearing in Eq.(3).Then we get $M = 2$. Thus, the trial solution of Eq.(3) can be stated as

$$u(\eta) = a_0 + a_1 e^{-\varphi(\eta)} + a_2 e^{-2\varphi(\eta)}, \tag{7}$$

where $a_2 \neq 0$, a_1 and a_2 are constants. Substituting u, along with (4) into ODE (3) and equating the coefficients of all powers of $\exp(-n\varphi)$ to zero, yields a set of algebraic equations for unknown coefficients a_n and parameters μ,λ,V. We solve the set of algebraic equations and depending on the relations between the parameters λ and μ we construct the exact travelling wave solutions for (1) initial PDE. Thus putting (7) in Eq.(3) and comparing, we get

$$\begin{cases} -Va_0 + (\delta - \beta V)(\lambda a_1 \mu + 2a_2\mu^2) + \alpha a_0^2 + c = 0, \\ -Va_1 + (\delta - \beta V)(2a_1\mu + 6a_2\lambda\mu + \lambda^2 a_1) + 2\alpha a_0 a_1 = 0, \\ -Va_2 + (\delta - \beta V)(3\lambda a_1 + 4a_2\lambda^2 + 8\mu a_2) + \alpha a_1^2 + 2\alpha a_0 a_2 = 0, \\ (\delta - \beta V)(2a_1 + 10a_2\lambda) + 2\alpha a_1 a_2 = 0, \\ 6(\delta - \beta V)a_2 + \alpha a_2^2 = 0. \end{cases} \tag{8}$$

Note that the analogous system of [4] is wrong. By solving the algebraic system (8), we define

$$\begin{cases} a_0 = \dfrac{1}{2\alpha}[V + (\beta V - \delta)(\lambda^2 + 8\mu)], \\ a_1 = 6\dfrac{\lambda}{\alpha}(\beta V - \delta) = \lambda a_2, \\ a_2 = \dfrac{6}{\alpha}\left(\beta V - \delta\right). \end{cases} \tag{9}$$

Note that the second equation of system (8) is automatically satisfied. Further, we assume that $\beta V - \delta \neq 0$. From the first equation of (8), we get

$$\left(\lambda^2 - 4\mu\right)^2 = \frac{V^2 - 4\alpha c}{\left(\beta V - \delta\right)^2}, \tag{10}$$

which means $V^2 > 4\alpha c$. Thus

$$\lambda^2 - 4\mu = \pm\frac{\sqrt{V^2 - 4\alpha c}}{|\beta V - \delta|}. \tag{11}$$

When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$ we consider $\lambda^2 - 4\mu = \dfrac{\sqrt{V^2 - 4\alpha c}}{|\beta V - \delta|}$ and the solution (7) becomes

$$u = \frac{1}{2\alpha}\left[V + (\beta V - \delta)(\lambda^2 + 8\mu)\right]$$
$$+\frac{12\mu(\beta V - \delta)}{\alpha}\cdot\frac{2\mu - \lambda^2 - \lambda\sqrt{\lambda^2 - 4\mu}\tanh\left[\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\eta + c)\right]}{\left\{\sqrt{\lambda^2 - 4\mu}\tanh\left[\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\mu + c)\right] + \lambda\right\}^2}. \tag{12}$$

The graph of this solution with chosen parameters is given in Fig.1 showing a row of solitons. Number of solitons is regulated by the value of V.

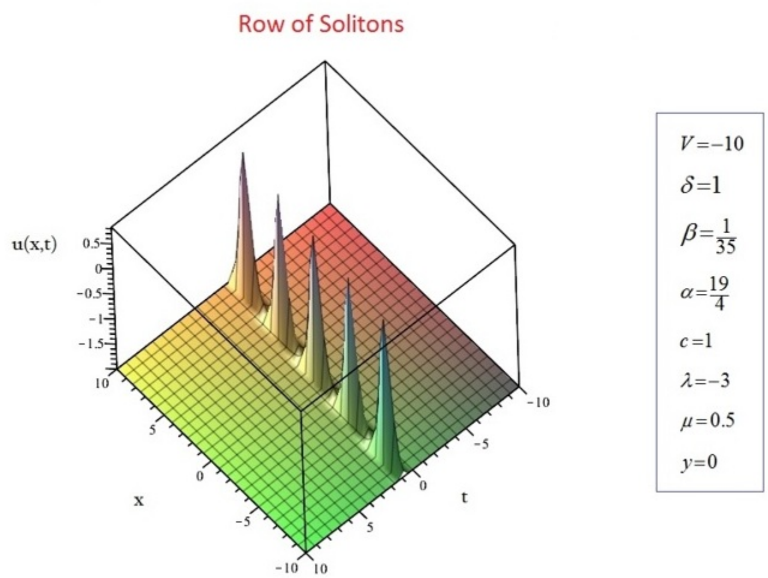

Figure 1:

3 Summary. The $\exp(-\varphi(\eta))$-expansion method has been successfully applied to find the exact solutions of the (2+1)D nonlinear ZK-MEW equation (1). The found solution (12) represents the solitary wave solution of solution forms, expressed, through the hyperbolic tangent functions. Solution (12) has been verified by substituting back into the original Eq. (1) and found correct. Revision of the previously received wrong solutions [4] is carried out.

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