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THE CONTACT PROBLEM FOR PIECEWISE-HOMOGENEOUS VISCOELASTIC PLATE WITH INCLUSION

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Abstract. A piecewise-homogeneous viscoelastic plate, reinforced with a semi-infinite rigid inclusion, which meets the interface at a right angle and is loaded with normal forces is considered. The normal contact stresses along the contact line are determined and the behavior of the contact stresses in the neighborhood of singular points is established. The problem is reduced to a singular integral equation of first kind with fixed singularity. Using the methods of the theory of analytic functions a Riemann problem is obtained, the solution of which is presented in an explicit form.

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1 Introduction.

The first fundamental problem for a piecewise-homogeneous plane was solved, when a crack of finite length arrives at the interface of two material at the right angle [1], and also similar problem for a piecewise-homogeneous plane when acted upon by symmetrical normal stresses at the crack sides[2, 3]. The contact problems for a piecewise-homogeneous planes with a semi-infinite and finite inclusion are investigated in [4, 5].

2 Statement and solution of the problem.

Suppose the body holds $z = x + iy$ complex plane, which consists of two dissimilar isotropic half-plane with viscoelastic property and it is reinforced with semi-infinite rigid inclusion, on the inclusion acts the normal load with intensity $p_0(x,t)$.

The half-planes $S_1 = \{z|\ \mathrm{Re}z > 0,\ z \notin [0, +\infty]\}$, $S_2 = \{z|\mathrm{Re}z < 0\}$ are connected along Oy axis. The contact conditions along the interface have the form:

$$\sigma_x^{(1)} = \sigma_x^{(2)},\ \tau_{xy}^{(1)} = \tau_{xy}^{(2)},\ \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y},\ \frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial y}. \tag{1}$$

On the boundary of interaction of rigid inclusion and half-plane S_1 the following conditions are valid:

$$\begin{aligned} &\sigma_y^{(1)+} - \sigma_y^{(1)-} = p(x,t),\ \ \tau_{xy}^{(1)+} - \tau_{xy}^{(1)-} = 0, \\ &u_+^{(1)} - u_-^{(1)} = 0,\ \ v_+^{(1)} = v_-^{(1)} = v(x,t), \end{aligned} \tag{2}$$

$$\frac{dv_0(x,t)}{dx} = 0, \quad \frac{dv_0(x,t)}{dx} = \frac{dv(x,t)}{dx}, \tag{3}$$

$$\int_0^\infty [p(x,t)-p_0(x,t)]dt=0, \tag{4}$$

where (2) represents the jumps of the stress and displacement components of the plate points on the contact line, (3) represents the constancy of the normal displacement $v_0(x,t)$ of the inclusion points and rigid contact condition between the plate and the inclusion, (4) is the equilibrium condition of the inclusion.

In the theory of viscoelatisity we have Kolosov-Muskhelishvili's type formulas [15,16]:

$$\sigma_y^{(k)}-i\tau_{xy}^{(k)}=\Phi^{(k)}(z,t)+\overline{\Phi^{(k)}(z,t)}+z\overline{\Phi'^{(k)}(z,t)}+\overline{\Psi^{(k)}(z,t)}, \tag{5}$$

$$(I-L)\left[\varkappa_k\Phi^{(k)}(z,t)-\overline{\Phi^{(k)}(z,t)}-z\overline{\Phi'^{(k)}(z,t)}-\overline{\Psi^{(k)}(z,t)}\right]=2\mu_k(u_k'+iv_k'), \tag{6}$$

where

$$(I-L)g_k(t)=g_k(t)-\int_{t_0}^t E_k\frac{\partial}{\partial\tau}C_k(t,\tau)g_k(\tau)d\tau,\quad 2\mu_k=\frac{E_k}{1+\nu_k}$$

and $\varkappa_k=\begin{cases}3-4\nu_k\\ \frac{3-\nu_k}{1+\nu_k}\end{cases}$ $k=1,2$. $C_k(t,\tau)$ and E_k are the creep measure and the Jung's module of the materials, respectively. Besides, the plate Poisson's coefficients for elastic-instant deformation $\nu_k(t)$ and creep deformation $\nu_k(t,\tau)$ are the same and constant: $\nu_k(t)=\nu_k(t,\tau)=\nu_k=const$.

From relations (5), (6) we obtain the following Riemann boundary value problems

$$\Phi_+^{(1)}(x,t)-\Phi_-^{(1)}(x,t)=\frac{1}{\varkappa_1+1}p(x,t),$$

$$\Psi_+^{(1)}(x,t)-\Psi_-^{(1)}(x,t)=\frac{\varkappa_1-1}{\varkappa_1+1}p(x,t)-\frac{1}{\varkappa_1+1}xp'(x,t),\quad x>0.$$

The general solutions of this problems will be represented as follows [17]:

$$\begin{aligned}&\Phi^{(1)}(z,t)=\tfrac{1}{2\pi(\varkappa_1+1)i}\int_0^\infty \tfrac{p(x,t)dx}{x-z}+W_1(z,t)\equiv A_1(z,t)+W_1(z,t),\\ &\Psi^{(1)}(z,t)=\tfrac{\varkappa_1-1}{2\pi(\varkappa_1+1)i}\int_0^\infty \tfrac{p(x,t)dx}{x-z}-\tfrac{1}{2\pi(\varkappa_1+1)i}\int \tfrac{xp'(x,t)dx}{x-z}+Q_1(z,t)\\ &\equiv B_1(z,t)+Q_1(z,t),\end{aligned}$$

where $W_1(z,t)$, $Q_1(z,t)$ are unknown analytic functions in the half-plane S_1, which will be defined from condition (1) on the interface.

Using the methods of the theory of analytic functions (particularly, using the Cauchy's theorems) the sought analytic functions are represented in following form

$$\begin{aligned}&W_1(z,t)=-e_1t_1\int_0^\infty \tfrac{xp'(x,t)dx}{x+z}+e_1t_1\int_0^\infty \tfrac{xp(x,t)dx}{(x+z)^2}+e_1t_1(\varkappa_1-1)\int_0^\infty \tfrac{p(x,t)dx}{x+z},\\ &\Phi^{(2)}(z,t)=h_3t_1\int_0^\infty \tfrac{p(x,t)dx}{x-z},\\ &Q_1(z,t)=-e_1t_1\int_0^\infty \tfrac{x^2p'(x,t)dx}{x+z}+m_1t_1\int_0^\infty \tfrac{p(x,t)dx}{x+z}+e_1t_1(\varkappa_1-1)\int_0^\infty \tfrac{xp(x,t)dx}{(x+z)^2}\\ &+e_1t_1z\int_0^\infty \tfrac{p(x,t)dx}{(x+z)^2}+2e_1t_1z\int_0^\infty \tfrac{xp(x,t)dx}{(x+z)^3},\\ &\Psi^{(2)}(z,t)=(h_3-h_4)t_1z\int_0^\infty \tfrac{p(x,t)dx}{(x-z)^2}-h_4t\int_0^\infty \tfrac{xp'(x,t)dx}{x-z}\\ &+(h_4(\varkappa_1-1)+m_1)t_1\int_0^\infty \tfrac{p(x,t)dx}{x-z},\end{aligned} \tag{7}$$

where $t_1 = \frac{1}{2\pi i(\varkappa_1+1)}$, $e_1 = \frac{\mu_2-\mu_1}{\varkappa_1\mu_2+\mu_1}$, $e_2 = \frac{\mu_2-\mu_1}{\varkappa_2\mu_2+\mu_1}$, $m_1 = (\varkappa_1+1)\left[\frac{1}{\varkappa_2\mu_1+\mu_2} - \frac{1}{\varkappa_1\mu_2+\mu_1}\right] = h_2 - h_4$, $m_2 = (\varkappa_2+1)\mu_1\left[\frac{1}{\varkappa_2\mu_1+\mu_2} - \frac{1}{\varkappa_1\mu_2+\mu_1}\right] = h_3 - h_1$, $h_1 = \frac{(\varkappa_2+1)\mu_1}{\varkappa_1\mu_2+\mu_1}$, $h_2 = \frac{(\varkappa_2+1)\mu_2}{\varkappa_2\mu_1+\mu_2}$, $h_3 = \frac{(\varkappa_2+1)\mu_1}{\varkappa_2\mu_1+\mu_2}$, $h_4 = \frac{(\varkappa_1+1)\mu_2}{\varkappa_1\mu_2+\mu_1}$. Therefore, based to the conditions (3), (4) and relations (7) for the semi-infinite rigid inclusion we get the following singular integral equation with fixed singularity

$$\int_0^\infty \frac{p(x,t)dx}{x-s} + e_1\varkappa_1 \int_0^\infty \frac{p(x,t)dx}{x+s} = 0,\ s>0 \tag{8}$$

$$\int_0^\infty [p(x,t) - p_0(x,t)]dx = 0, \tag{9}$$

where $p_0 \in H$ is integrable function on the arbitrary finite interval of $[0,\infty)$ and equal to zero outside some interval [18]. The solution of the problem (8), (9) is sought in the class of functions $p \in H^*$ on the arbitrary finite interval of $(0,\infty)$ and $p(x,t) = O(x^{-(1+\omega)}),\ x \to \infty,\ \omega > 0$.

Introducing the notation $\psi(x,t) = \int_0^x [p(y,t) - p_0(y,t)]dy$ making the substituting $x = e^\zeta$, $s = e^\xi$ and using generalized Fourier transform [19] with variable $\tau = \tau_0 + i\varepsilon$, we obtain the following equation

$$\left(\pi cth\pi\tau - \frac{\pi e_1\varkappa_1}{sh\pi\tau}\right)\tau\Psi(\tau,t) = A(\tau,t),\ |\tau_0| < \infty \tag{10}$$

where $\Psi(\tau,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \psi_0(\xi,t)e^{i\xi\tau}d\xi$, $A(\tau,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty P(\xi,t)e^{i\xi\tau}d\xi$, $\psi_0(\xi,t) = \psi(e^\xi,t)$, $P(\xi,t) = P_0(e^\xi,t)e^\xi$, $P_0(s,t) = -\int_0^\infty \frac{p_0(x,t)dx}{x-s} - e_1\varkappa_1\int_0^\infty \frac{p_0(x,t)dx}{x+s}$, $P(\xi,t) = O(e^\xi),\ \to -\infty$, $P(\xi,t) = O(1),\ \to \infty$.

The function $A(w,t)$ is analytic in the strip $-\varepsilon < \mathrm{Im}w < 1-\varepsilon$, vanish at infinite as $|w|^{-1}$, has the poles of the first order in the points $w = 0,\ w = i$. Therefore, based on equation (10) the function

$$w\Psi(w,t) = \frac{A(w,t)sh\pi w}{\pi(ch\pi w - e_1\varkappa_1)}$$

is continuously extendable on the boundaries of the strip $0 < \mathrm{Im}w < 1$, analytic in this strip, with the exception of the points w_i $(i = 0,1,2,3,...)$, that are the roots of the equation

$$E(w) = ch\pi w - e_1\varkappa_1 = 0,\ |e_1\varkappa_1| < 1. \tag{11}$$

Let $w_0 = iy_0$ $(y_0 = \frac{1}{\pi}\arccos(e_1\varkappa_1)$ $0 < y_0 < 1)$ are the roots of equation (11) in the strip $-1 < \mathrm{Im}w < 1$. Using the inverse Fourier transform and Cauchy's theorem about residue to the function $w\Psi(w)e^{-iw\xi}$ we obtain the solution of problem (8), (9) in the form

$$\psi'(x,t) \equiv p(x,t) - p_0(x,t) = \frac{x^{-1}}{\sqrt{2\pi}}\lim_{\rho\to 0}\int_{-\infty}^\infty is\Psi(s,t)e^{-\varrho|s|}e^{-is\ln x}ds, \tag{12}$$

which satisfies the following estimates at singular points

$$\psi'(x,t) = K_1(t)x^{y_0-1} + \alpha(t)O(x^{\beta-1}),\ x \to 0+, \tag{13}$$

$$\psi'(x,t) = K_2(t)O(x^{-\tilde{y}-1}),\ x \to +\infty, \tag{14}$$

where $\tilde{y} = \min(y_0, y_1)$, $A(x_1 - iy_1) = 0,\ y_1 > 0$.

The obtained results can be formulated as

Theorem. *problem (8), (9) has the solution, which is represented effectively by (12) and admits estimates (13), (14).*

Remark. For estimates (13) ,(14) the following conclusions are valid

a) If $e_1 > 0$, $(\mu_2 > \mu_1)$, then $0 < y_0 < 1/2$,

b) If $e_1 > 0$, $(\mu_2 < \mu_1)$, then $1/2 < y_0 < 1$,

c) If $e_1 = 0$, $(\mu_2 = \mu_1)$, then $y_0 = 1/2$,

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