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ON ONE-DIMENSIONAL NONLINEAR SYSTEM BASED ON MAXWELL MODEL

Mikheil Gagoshidze

Abstract. The initial-boundary value problem for one-dimensional system of nonlinear partial differential equations with the mixed boundary condition is considered. It is proved that in some cases of nonlinearity there exists a critical value ψ_c of the boundary data such that for $0 < \psi < \psi_c$ the steady state solution of the studied problem is linearly stable, while for $\psi > \psi_c$ is unstable. It is shown that as ψ passes through ψ_c then the Hopf type bifurcation may take place.

Keywords and phrases: Nonlinear partial differential one-dimensional system, stationary solution, linear stability, Hopf bifurcation.

AMS subject classification (2010): 35B32, 35B35, 35Q61.

The present note deals with a nonlinear model which is obtained after adding two terms to the second equation of well-known Maxwell's system in one-dimensional case [14]. This model is also some generalization of a system with two partial differential equations describing many other processes (see, for instance, [1], [9], [12] and references therein).

In the cylinder $[0,1]\times[0,\infty)$, let us consider the following initial-boundary value problem:

$$\begin{gathered}
\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V^{\alpha}\frac{\partial U}{\partial x}\right),\\
\frac{\partial V}{\partial t}=-aV^{\beta}+bV^{\gamma}\left(\frac{\partial U}{\partial x}\right)^{2}+cV^{\delta}\frac{\partial U}{\partial x},\\
U(0,t)=0,\quad V^{\alpha}\frac{\partial U}{\partial x}\bigg|_{x=1}=\psi,\\
U(x,0)=U_0(x),\quad V(x,0)=V_0(x).
\end{gathered}\tag{1}$$

Many works are dedicated to the investigation and numerical solution of (1) type models (see, for example, [2]-[13]). Here t and x are time and space variables respectively, $U=U(x,t)$, $V=V(x,t)$ are unknown functions, U_0, V_0 are given functions, and $a,b,c,\alpha,\beta,\gamma,\delta,\psi$ are known positive parameters.

If $\delta=\gamma-\alpha$, it is easy to check that the unique stationary solution of problem (1) is

$$(U_s,V_s)=\left(\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{-\alpha}{2\alpha+\beta-\gamma}}\psi x,\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}}\right).\tag{2}$$

Introducing a notation $W = V^{\alpha}\dfrac{\partial U}{\partial x}$, after simple transformations, we get

$$
\begin{gathered}
\frac{\partial W}{\partial t} = V^{\alpha}\frac{\partial^2 W}{\partial x^2} + \alpha\left(-aV^{\beta-1} + bV^{\gamma-2\alpha-1}W^2 + cV^{\gamma-2\alpha-1}W\right)W, \\
\frac{\partial V}{\partial t} = -aV^{\beta} + bV^{\gamma-2\alpha}W^2 + cV^{\gamma-2\alpha}W, \\
\left.\frac{\partial W}{\partial x}\right|_{x=0} = 0, \quad W(1,t) = \psi,
\end{gathered} \tag{3}
$$

$$
W(x,0) = V_0^{\alpha}\frac{\partial U_0(x)}{\partial x}, \quad V(x,0) = V_0(x).
$$

The unique stationary solution of problem (3) is

$$
(W_s, V_s) = \left(\psi, \left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}}\right).
$$

Let

$$
W(x,t) = W_s(x) + W_1(x)e^{\lambda t} = \psi + W_1(x)e^{\lambda t},
$$

$$
V(x,t) = V_s(x) + V_1(x)e^{\lambda t} = \left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}} + V_1(x)e^{\lambda t}.
$$

We investigate the linear stability of problem (3) by linearizing near the stationary solution (W_s, V_s). After some transformations we have:

$$
\begin{gathered}
\frac{d^2 W_1(x)}{dx^2} + \eta^2 W_1(x) = 0, \\
\left.\frac{dW_1(x)}{dx}\right|_{x=0} = W_1(1) = 0,
\end{gathered} \tag{4}
$$

where

$$
\begin{gathered}
\eta^2 = \alpha a\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-\alpha-1}{2\alpha+\beta-\gamma}} + b\psi^2\left(\frac{b}{a}\psi^2 + \frac{b}{a}\psi\right)^{\frac{\gamma-3\alpha-1}{2\alpha+\beta-\gamma}} \\
-\alpha a(2\alpha+\beta-\gamma)(2b\psi+c)\psi\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta+\gamma-3\alpha-2}{2\alpha+\beta-\gamma}} \\
\times\left(\lambda + a(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}\right)^{-1} - \lambda\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{-\alpha}{2\alpha+\beta-\gamma}}.
\end{gathered}
$$

It is not difficult to show that problem (4) has nontrivial solutions if and only if

$$\eta^2=\eta_n^2=\left(n+\frac{1}{2}\right)^2\pi^2, \quad n\in Z_0.$$

For corresponding $\lambda=\lambda_n$ we have:

$$\lambda_n^2-P_n(\psi,\alpha,\beta,\gamma,a,b,c)\lambda_n+L_n(\psi,\alpha,\beta,\gamma,a,b,c)=0,$$

$$P_n(\psi,\alpha,\beta,\gamma,a,b,c)=\left(n+\frac{1}{2}\right)^2\pi^2\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\alpha}{2\alpha+\beta-\gamma}}$$

$$+a\left(\alpha+\beta-\gamma\right)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-1}{\alpha+\beta-\gamma}}-\alpha b\psi^2\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\gamma-2\alpha-1}{2\alpha+\beta-\gamma}},$$

$$L_n(\psi,\alpha,\beta,\gamma,a,b,c)=a\left(n+\frac{1}{2}\right)^2\pi^2(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}$$

$$-\alpha a\left(2\alpha+\beta-\gamma\right)\left[a\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{2(\beta-1)-\alpha}{2\alpha+\beta-\gamma}}+b\varphi^2\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta+\gamma-3\alpha-2}{2\alpha+\beta-\gamma}}\right].$$

Let us note that the stationary solution (W_s,V_s) of problem (3) is linearly stabile if and only if $Re(\lambda_n)<0$, for all n and unstable if there exists an integer m such that $Re(\lambda_m)>0$. From (4) it can be deduced that if $2\alpha+\beta-\gamma>0$, then stationary solution (W_s,V_s) of problem (3) is linearly stable if and only if $P_n(\psi,\alpha,\beta,\gamma,a,b,c)<0$, for all n, i.e.,

$$a\left(\gamma-\alpha-\beta\right)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-\alpha-1}{\alpha+\beta-\gamma}}+\alpha b\psi^2\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\gamma-3\alpha-1}{2\alpha+\beta-\gamma}}<\frac{\pi^2}{4}.$$

We studied the stability of the steady state solution which depends on a boundary condition $\psi>0$. For sufficiently small values of ψ the steady state solution is linearly stable. But as ψ passes through a critical value ψ_c, the stability changes and a Hopf bifurcation may take place [15].

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Author(s) address(es):

Mikheil Gagoshidze
I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: MishaGagoshidze@gmail.com, Mikheil.Gagoshidze@tsu.ge