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# AMERICAN OPTION PRICING IN MULTIDIMENSIONAL FINANCIAL MARKET

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**Abstract**. Financial $(B, S)$ market in discrete time with $k$ number of bonds and one risky asset is considered. Interest rate is introduced, which is the combination of interest rates $r_1, r_2, , r_k$ related to bonds. In this scheme, for the American option, representations of fair price, optimal stopping moment and hedging strategy are obtained.

**Keywords and phrases**: Financial market, American option, fair price, optimal stopping time, minimal hedge.

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**1 Model.** Let us consider the binomial financial $(B, S)$-market with one stock and $k$ number of bonds, with prices given by the following recurrent formulas

$$B_n^{(i)} = (1 + r_n^{(i)})B_{n-1}^{(i)}, \tag{1}$$

$$S_n = (1 + \rho_n)S_{n-1}, \tag{2}$$

$i = 1, 2, ..., k$, $1 \le n \le N$, where $r_n^{(i)} > -1$ are the interest rates and $\rho_n$ is a sequence of independent identically distributed random variables, $B_0^{(i)}$ and $S_0$ are deterministic. $\rho_n$ takes only two possible values $a$ and $b, -1 < a < b$, with probabilities $1 - p$ and $p(0 < p < 1)$ respectively [2], [3].

Suppose, that $B_n$ presents the sum of $B_n^{(i)}$ bonds

$$B_n = \sum_{i=1}^{k} B_n^{(i)} = \sum_{i=1}^{k} (1 + r_n^{(i)})B_{n-1}^{(i)}. \tag{3}$$

This is quite natural, since it represents the total capital in bonds at the moment $n$. Then we can calculate interest rate associated to such $B_n$, which makes same effect as $k$ number of bonds. So, that from (1),(3) we have

$$(1 + r_n) \sum_{i=1}^{k} B_{n-1}^{(i)} = \sum_{i=1}^{k} (1 + r_n^{(i)})B_{n-1}^{(i)},$$

and it follows immediately, that interest rate $r_n$ of $B_n$ is given by the formula

$$r_n = \frac{\sum_{i=1}^{k} r_n^{(i)} B_{n-1}^{(i)}}{\sum_{i=1}^{k} B_{n-1}^{(i)}}. \tag{4}$$

**2 Content.** It follows immediately from (1), (2), (4) that we have financial market with two actives that satisfies the following discrete stochastic differential equations

$$\Delta B_n = r_n B_{n-1}, \tag{5}$$

$$\Delta S_n = \rho_n S_{n-1}. \tag{6}$$

Note, that in this scheme market is complete and using equality (4) unique risk-neutral probability measure one can constructed [2]

$$p^* = \frac{r_n - a}{b - a},$$

and the equality $E^* \rho_n = r_n$, is valid where $E^*$ denotes expectation by the measure $P^*$. So $\frac{S_n}{B_n}, n = 0, 1, ..., N$ represents the martingale with respect to the measure $P^*$.

Suppose now that the American option with pay-off function $f = (f_n), n = 0, 1, , N$ is given, as a non-negative stochastic sequence. There three main problems in valuation of American option, determine minimal price of option that gives possibility to execute contingent claim $f$, define optimal stopping time and construct minimal hedging strategy.

Below we will assume that filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$ is given, where $\mathcal{F}_n = \sigma\{S_0, ..., S_n\}$ is the minimal $\sigma$ -algebra subtended by $S_0, ..., S_n$.

Let $X_0 = x > 0$, be the initial amount of investor and the capital related to the self-financing portfolio $\pi_n = (\beta_n, \gamma_n)$ at the moment $n$ is

$$X_n^\pi = \beta_n B_n + \gamma_n S_n,$$

where $\beta_n$ and $\gamma_n$ are the quantities of bonds and stocks, here $\gamma_n$ are $\mathcal{F}_{n-1}$ measurable.

Portfolio $\pi_n = (\beta_n, \gamma_n)$ is called American type $(x, f, N)$ hedge if for any $\omega \in \Omega$

$$X_0^\pi(\omega) = x, \; X_n^\pi(\omega) \geq f_n(\omega), \;\; n \leq N.$$

In this case for any stopping moment $\tau \leq N$ the inequality $X_\tau \geq f_\tau$ is true. The hedge is minimal if there is a stopping moment $\tau$ such that $X_\tau^\pi(\omega) = f_\tau(\omega)$, for any $\omega \in \Omega$.

Denote by $\Pi(x, f, N)$ the set of $(x, f, N)$ hedges. The value

$$C_N = \inf\{x > 0 : \Pi(x, f, N) \neq \emptyset\}$$

is called the fair price of the option.

Consider the sequence

$$Y_n = \sup_{n \leq \tau \leq N} E^* \left( \frac{f_\tau}{B_\tau} / \mathcal{F}_n \right).$$

It is not difficult to check that

$$Y_N = \frac{f_N}{B_N}, \;\; Y_n = \max\left\{ \frac{f_n}{B_n}, E^*(Y_{n+1}/\mathcal{F}_n) \right\}.$$

Then it is clear that

$$Y_n \geq E^*(Y_{n+1}/\mathcal{F}_n),\ Y_n \geq \frac{f_n}{B_n},\ n \leq N-1,$$

so, $Y_n$ is a supermartingale and since $(B,S)$ market defined by (5),(6)is complete, then $Y_n$ admits the following representation [2]

$$Y_n = Y_0 + \sum_{k=1}^{n} \frac{\gamma_k S_{k-1}}{B_k}(\rho_k - r_k) - A_n, \tag{7}$$

where $A_n$ is a nondecreasing predictable stochastic process ($A_0 = 0$).

According to the general theory of optimal stopping rules, the following stopping time is optimal in class $n \leq \tau \leq N$

$$\tau_n^* = \min\left\{n \leq k \leq N : Y_k = \frac{f_k}{B_k}\right\}.$$

It means that

$$E^*\left(\frac{f_{\tau_n^*}}{B_{\tau_n^*}}/\mathcal{F}_n\right) = \sup_{n\leq\tau\leq N} E^*\left(\frac{f_\tau}{B_\tau}/\mathcal{F}_n\right).$$

In particular since $\mathcal{F}_0 = \{\emptyset, \Omega, \}$, for $\tau = \tau_0^*$ it follows that

$$Y_0 = \sup_{0\leq\tau\leq N} E^* \frac{f_\tau}{B_\tau}.$$

Thus the following theorem is valid

**Theorem.** *Let financial market $(B,S)$ be given by (5), (6) recurrent equalities. Then for the American contingent claim with payoff $f = (f_n), n = 0, 1, , N$*

*1) the fair price of option*

$$C_N = \sup_{0\leq\tau\leq N} E^* \varepsilon_\tau^{-1}(U) f_\tau,\ \ U_n = \sum_{k=1}^{n} r_k$$

*where $\varepsilon_n(U) = \prod_{k=1}^{n}(1 + \Delta U_k)$.*

*2) optimal stopping time*

$$\tau_n^* = \min\left\{0 \leq k \leq N : Y_k = \frac{f_k}{B_k}\right\},$$

*3) minimal hedging strategy $\pi^* = (\beta_n^*, \gamma_n^*)$*

$$\gamma_n^* = \frac{\gamma_n S_{n-1}}{B_n} \quad \beta = \frac{X_{n-1}^{\pi^*} - \gamma_n^* S_{n-1}}{B_{n-1}},$$

*where $\gamma_n$ is the predictable sequence form representation (7).*

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