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ON MENSHOV-RADEMACHER THEOREM IN THE QUASI-ORTHOGONAL CASE *

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Abstract. We discuss the validity of Menshov-Rademacher theorem for quasi-orthogonal series introduced by D.A. Menshov in a paper published in 1927 and for quasi-orthogonal sequence considered in a paper of 1948 by M. Kac, R. Salem and A. Zygmund.

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1 Introduction. The following theorem, one of the key results for the theory of orthogonal series, was obtained independently in Menshov-1923=*Menchoff D.* "Sur les series de fonctions orthogonales I. Fund. Math., 1923, V. 4, P. 82-105" and in Rademacher-1922=*Rademacher H.*, Einige Satze uber Reiheen von allgemeinen Orthogonalfunktionen. Math. Ann. 1922, Bd. 87, 1-2, S. 112-138 (Received : 08 October 1921).

Theorem 1. *(Menshov-Rademacher). Let $\varphi_n \in L_2[0,1],\ n = 1, 2, \ldots$ be an orthonormal sequence and $a_n, n = 1, 2, \ldots$ let be a sequence of real (or complex) numbers such that*

$$\sum_{n=1}^{\infty} |a_n|^2 \log_2^2 n < \infty. \qquad (MR)$$

Then the series $\sum_n a_n\varphi_n$ converges a.e.

According to [10], Menshov had proved Theorem 1 already in the summer of 1920. We note that Menshov-1923 contains also the first proof of the *delicate assertion* that **not every** orthonormal sequence $\varphi_n \in H = L_2[0,1],\ n = 1, 2, \ldots$ is a **convergence system**, i. e., the condition

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty \qquad (SQ)$$

is not sufficient for the validity of Theorem 1.

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Note that in 1922 a very important example of an orthonormal sequence $r_n : [0,1] \to \{1,-1\}, n = 1,2,\dots$ was consttructed by Rademacher (now called *the sequence of Rademacher functions*) which is a convergence system. Later on in *A. Kolmogoroff and D. Menchoff*, "Sur la convergence des series des fonctions orthogonales, Math. Zeitschrift, 26 (1927), 432-441" it was shown that **not every orthonormal** sequence $\varphi_n : [0,1] \to \{1,-1\}, n = 1,2,\dots$ is a convergence system.

Theorem 1 is included in 1935's monograph [7] with a proof based on Menshov-Rademacher inequality (see Russian version of [7], p. 292, Lemma 1; see also, [3, 4, 5]).

In Kantorowitch-1937=*L. B. Kantorovich, "Some theorems on the almost everywhere convergence, Dokl. Akad. Nauk. SSSR, 14 (1937), 537-540"a proof of the following version of Theorem 1 is outlined:*

Theorem 1K*(Kantorovich). Let $\varphi_n \in H = L_2[0,1]$, $n = 1,2,\dots$ be an orthonormal sequence and $a_n, n = 1,2,\dots$ let be a sequence of numbers satisfying (MR). Then the series $\sum_n a_n\varphi_n$ converges a.e. and the estimate*

$$\int_0^1 \left(\sup_{n\geq 1} |\sum_{k=1}^n a_k\varphi_k(t)|\right)^2 dt \leq K \sum_{n=1}^\infty |a_n|^2(1+\log_2 n)^2$$

holds with some universal constant K.

Kantorowitch-1937 and its result are mentioned in [1], while Theorem 1K is included in [8] without mentioning Kantorowitch-1937.

Different proofs of Theorem 1 can be seen in Salem-1941=*R. Salem*, "A new proof of a theorem of Menchoff, Duke Mathematical Journal, 8 (1941), 269-272" and in SuYa-1950=*G.I. Sunouchi, S. Yano*, "Notes on Fourier Analysis (XXXIX). Convergence and Summability of Orthogonal Series, Proceedings of the Japan Academy, 26(1950), 10-16".

The proof of SuYa-1950 is used in [1]. In the next sections we shall discuss quasi-orthogonal generalizations of Theorem 1 and Theorem 1K.

2 Menshov's quasi-orthogonality. In Menshov-1927=*Menchoff D. A.*, "Sur les series des fonctiones orthogonales III. Fund. Math., 10 (1927), 375-420 (is submitted on July 15, 1923)" it is introduced a notion of a quasi-orthogonal series as follows.

For numbers $a_n, n = 1,2,\dots$ and for functions $\varphi_n \in L_2[0,1]$, $\int_0^1 |\varphi_n(t)|^2 dt = 1$, $n = 1,2,\dots$ the series

$$\sum_n a_n\varphi_n$$

is called *quasi-orthogonal* if there exists a convergent series of positive terms

$$\sum_n \rho_n^2$$

such that

$$\|\sum_{i=n}^{n+p} a_i\varphi_i\|^2 \leq \sum_{i=n}^{n+p} \rho_i^2,\ n = 1,2,\dots,\ p = 0,1,2\dots \qquad (MQO)$$

Such a strange definition! Menshov's quasi-orthogonality is not considered in [2, 6], where the closely related concepts are treated.

Theorem $1'$ *(Menshov) Let*

$$\sum_n a_n\varphi_n \qquad (QOS)$$

be a quasi-orthogonal series for which the sequence $\rho_n, n = 1, 2, \dots$ from (MQO) satisfies the (MR) condition. Then (QOS) converges a.e.

In Menshov-1927 instead of proving Theorem $1'$ the following is stated: if we analyze proof of results of Weyl, Hobson, Plancherel and proof of Theorem 1, we conclude that Theorem $1'$ can be proved as well.

Let us recall the mentioned results and hence a pre-history of Theorem 1.

H. Weyl in 1909 proved a variant of Theorem 1 with the sequence $(\sqrt{n})$ instead of $(\log_2^2 n)$; *E.W. Hobson* in 1913 proved a variant of Theorem 1 with the sequence (n^ε) for some $\varepsilon > 0$ instead of $(\log_2^2 n)$; *Michel Plancherel* in Plancherel-1913="Sur la convergence des series de fonctions orthogonales. C.R. Acad. Sci. Paris. 1913. V. 157. P. 539-542" proved a version Theorem 1 with the sequence $(\log_2^3 n)$ instead of $(\log_2^2 n)$.

We note that in Menshov-1923 only Plancherel-1913 is mentioned.

A mentioning of Theorem $1'$ we found in Gaposhkin-1975=*V.F. Gaposhkin*, "Convergence of series connected with stationary sequences. Izv. AN SSSR, Ser. Matem., tom 39, vipusk 6, 1366-1392. English translation: Mathematics of the USSR-Izvestiya, Volume 9, Number 6".

3 KSZ's quasi-orthogonality.

M. Kac, R. Salem and A. Zygmund in KSZ-1948="A gap theorem. Transactions of the American Mathematical Society, 1948, 63, 235-248" a notion of a quasi-orthogonal sequence was defined and shown that *a sequence $\varphi_n \in L_2[0,1]$, $n = 1, 2, \dots$ is quasi-orthogonal in their sense if and only is the series $\sum_n a_n\varphi_n$ converges in $L_2[0,1]$ for every sequence (a_n) of numbers such that* $\sum_{n=1}^\infty |a_n|^2 < \infty$.

They wrote: "Let us now point out a few simple consequences of quasi-orthogonality, which do not seem to have been stated...in particular, Menchoff's theorem, asserting the convergence almost everywhere of $\sum_n a_n\varphi_n$ provided $\sum_{n=1}^\infty a_n^2 \log_2^2 n < \infty$ holds."

Now we show that this claim is correct. The way of derivation of the following assertion from Theorem 1K seems to be new.

Theorem 2. *Let $\varphi_n \in H = L_2[0,1], n = 1, 2, \dots$ be a quasi-orthogonal in the sense of KSZ-1948 sequence and let $a_n, n = 1, 2, \dots$ be a sequence of numbers satisfying(MR). Then*

$$\|(\varphi_n)\|_{2,w} := \sup\left\{\sum_{k=1}^\infty |\int_0^1 \varphi_k(t)\bar{\psi}(t)dt| : \psi \in L_2[0,1],\ \int_0^1 |\psi(t)|^2 dt \le 1\right\} < \infty\,, \quad (NE)$$

the series $\sum_n a_n\varphi_n$ converges a.e. and the estimate

$$\int_0^1 \left(\sup_{n\ge1} |\sum_{k=1}^n a_k\varphi_k(t)|\right)^2 dt \le C\|(\varphi_n)\|_{2,w}^2 \sum_{n=1}^\infty |a_n|^2(1 + \log_2 n)^2$$

holds with some universal constant C (it can be taken $C = 2K$ in the real case and $C = 4K$ in the complex case).

Proof. A proof of (NE) can be made in a standard way using the Closed Graph Theorem. We can suppose without loss of generality that $\|(\varphi_n)\|_{2,w} = 1$. Then according to [9, Proposition 2.1] we can find four orthonormal sequences $(\varphi_n^{(j)})\, j = 1, 2, 3, 4$ in $L_2[0, 1]$ such that

$$\varphi_n = c\left(\varphi_n^{(1)} + \varphi_n^{(2)} + \varphi_n^{(3)} + \varphi_n^{(4)}\right),\, n = 1, 2, \ldots \qquad (TTV)$$

where $c = \frac{1}{\sqrt{2}}$ in the real case and $c = \frac{1}{2}$ in the complex case.

Now from (TTV) and from application of Theorem 1K to each of summands in (TTV) we get that Theorem 2 is true. □

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