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ON ONE NONLINEAR INTEGRO-DIFFERENTIAL PARABOLIC EQUATION

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Abstract. The fourth-order integro-differential parabolic equation is considered. The stability and uniqueness of the solution to one initial-boundary value problem is given.

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The mathematical description of many processes is carried out by integro-differential equations and their systems. Such models have been arisen in connection to the local partial derivative equations. Nevertheless, the study of these models began relatively late. The models of integro-differential type discussed in the presented work were first proposed in [5]. They arose, on the one hand, while describing real diffusion processes (see, for instance, [1], [2], [7], [9], [10], [13], [14] and references therein), and on the other hand, during the generalization of well-known nonlinear parabolic equations, to which many scientific works are devoted to (see, for instance, [4], [12] and references therein). A characteristic feature of these models is that nonlinear coefficients depend on higher-order derivatives that contain integrals with respect to time variables of the derivatives of functions (solutions) we are searching for.

As already mentioned, the reduction of the Maxwell system of differential equations to the form of integro-differential equations was first performed in [5] and has the following form

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int\limits_0^t |rotH|^2 d\tau\right) rotH\right], \tag{1}$$

where, $H = (H_1, H_2, H_3)$ is the vector of the magnetic field.

The above-mentioned integro-differential model (1) is complicated and so far, can be studied only for particular classes of nonlinearity (see, for instance, [3], [5] - [9], [11] and references therein).

The presented work discusses a natural mathematical generalization of the scalar analog of the integro-differential model (1). In particular, the corresponding fourth-order integro-differential equation is investigated. The stability and uniqueness of the solution to one initial-boundary value problem is studied.

In the rectangle $Q_T = [0,1] \times [0,T]$, where T is a positive constant, the following initial-boundary value problem is considered:

$$\frac{\partial u}{\partial t} + \frac{\partial^2}{\partial x^2}\left\{\left[1 + \int\limits_0^t \left(\frac{\partial^2 u}{\partial x^2}\right)^2 d\tau\right]\frac{\partial^2 u}{\partial x^2}\right\} = f(x,t), \tag{2}$$

$$u(0,t) = u(1,t) = 0, \tag{3}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \tag{4}$$

$$u(x,0) = u_0(x). \tag{5}$$

In equation (2) and in the initial condition (5), f and u_0 are given functions of their arguments.

It is easy to show that the solution to problem (2) - (5) is stable with respect to the right-hand side f and the initial condition u_0.

Multiplying equation (2) by the function u, integrating on $[0;1]$, using the formula of the integration by partial, and applying the Poincare-Friedrichs inequality

$$\int_0^1 u^2 dx \le \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx \le \int_0^1 \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx,$$

the following estimation is easily obtained

$$\|u\|^2 \le \int_0^t \|f(\tau)\|^2 d\tau + \|u_0\|^2,$$

where $\|\cdot\|$ denotes the norm of the space $L_2(0,1)$. The last inequality means the stability of the solution of problem (2) - (5) with respect to the right side f and the initial condition u_0.

Now, let us turn to the question of uniqueness of the solution. Suppose that u_1 and u_2 are two solutions of problem (2) - (5). If $w = u_1 - u_2$, then we have

$$\frac{\partial w}{\partial t} + \frac{\partial^2}{\partial x}\left\{\left[1 + \int_0^t \left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 d\tau\right]\frac{\partial^2 u_1}{\partial x^2} - \left[1 + \int_0^t \left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 d\tau\right]\frac{\partial^2 u_2}{\partial x^2}\right\} = 0, \tag{6}$$

$$w(0,t) = w(1,t) = 0, \tag{7}$$

$$\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(1,t) = 0, \tag{8}$$

$$w(x,0) = 0. \tag{9}$$

Let us multiply equation (6) by w and integrate the obtained equation by $[0,1]$. If we use the formula of integration by parts twice, boundary conditions (7), (8), and the easily verifiable inequality

$$(ca - db)(a - b) \ge \frac{1}{2}(c - d)(a^2 - b^2),$$

assuming that

$$c = \int_0^t \left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 d\tau, \quad d = \int_0^t \left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 d\tau, \quad a = \frac{\partial^2 u_1}{\partial x^2}, \quad b = \frac{\partial^2 u_2}{\partial x^2},$$

we will have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{1}{2}\int\limits_0^1\int\limits_0^t\left[\left(\frac{\partial^2 u_1}{\partial x^2}\right)-\left(\frac{\partial^2 u_2}{\partial x^2}\right)\right]d\tau\cdot\left[\left(\frac{\partial^2 u_1}{\partial x^2}\right)-\left(\frac{\partial^2 u_2}{\partial x^2}\right)\right]dx\le 0.$$

Using the following notation

$$\varphi(x,t)=\int\limits_0^t\left[\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2-\left(\frac{\partial^2 u_2}{\partial x^2}\right)^2\right]d\tau,$$

finally we arrive at

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{1}{4}\frac{d}{dt}\int\limits_0^1\varphi^2dx\le 0.$$

After integrating over t and taking into account the initial condition (9), we obtain

$$\|w\|^2\le 0.$$

From the last inequality it follows that $\|w\|=0$, which proves the uniqueness of the solution of problem (2)-(5).

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