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MATHEMATICAL MODELING OF EXPLOSIVE PROCESSES IN NONHOMOGENEOUS GRAVITATING GAS BODIES

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Abstract. The work considers a nonautomodel problem about the central explosion of nonhomogeneous gas body (star) bordering vacuum, which is in equilibrium in its own gravitational field. Asymptotic method of thin impact layer is used to solve the problem. The solution of the problem in the vicinity behind the shock wave (discontinuity surface of the first kind) is sought in the form of a singular asymptotic decomposition by a small parameter. Analytically, the main approximation for the law of motion and the thermodynamic characteristics of the medium was accurately found. The Cauchy problem for zero approximation of the law of motion of the shock wave is solved exactly, in the form of elliptic integrals of the first and second kind. Corresponding asymptotics of solutions are calculated.

Keywords and phrases: Nonhomogeneous body, gravitational field, explosion, shock wave, singular decomposition.

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1 Introduction. To solve many problems of astrophysics theory, it is necessary to study the dynamics of the interaction of gas bodies with the gravitational field. The concept of studying celestial phenomena should be based on posing and solving a number of dynamic problems about the motion of gravitational gas, which are considered to be important mathematical models of stellar motion and evolution [1]. The most essential and practically important parameter in these tasks is the law of shock wave motion generated by the explosion, but the classical formulation of the task in the language of partial derivatives differential equations usually involves the predetermination of the properties of the whole local process. Earlier, we derived a system of integrodifferential inequalities for one-dimensional spherical-symmetric motion of a perfect gravitating gas based on the equations of motion of the medium, integral equations of energy and Lagrange-Jacobi, to determine the law of motion of the shock or detonation wave and the moment of inertia of the perturbed area [2]. Then we solved several one-dimensional and two-dimensional nonautomodel problems about explosive processes in gas bodies and the spread of a detonation or shock wave to the surface of the body, followed by a separation into the vacuum [3–12]. In these mathematical models, we considered mainly homogeneous gas bodies.

2 Statement of the mixed problem for the system of nonlinear equations in partial derivatives. We will use the equations of the adiabatic spherical-symmetric motion of the gravitating gas in the Lagrangian form [2]:

$$\frac{\partial^2 r}{\partial t^2} + 4\pi r^2 \frac{\partial p}{\partial m} + \frac{km}{r^2} = 0,\ p = (\gamma-1)f(m)\rho^\gamma,\ \rho = [4\pi r^2 \frac{\partial r}{\partial m}]^{-1}, \tag{1}$$

here m is the mass of the $r(m,t)$ radius sphere, k is a gravity constant, γ is an adiabatic index, $f(m)$ function is associated with the distribution of entropy over the Lagrangian m coordinate, the $r = r(m,t)$ function defines the law of motion of the medium, $\partial r/\partial t$ is medium speed, $p(m,t)$ is medium pressure, $\rho(m,t)$ is medium density.
The first equation of system (1) is the equation of motion of the medium, the second equation is the adiabation equation, the third equation is the mass continuity equation, $r(m,t), p(m,t), \rho(m,t)$ are unknown functions.
The integral energy equation for the gas layer enclosed between the surfaces $m = 0$ and $m = M(t)$ has the form:

$$\begin{gathered} T + U + V = E + \int_0^t [\dot{M}(\tfrac{1}{2}(\tfrac{\partial r}{\partial t})^2 + \tfrac{p}{(\gamma-1)\rho} - \tfrac{kM}{R}) - 4\pi r^2 \tfrac{\partial r}{\partial t} p]_1 d\tau, \\ T = \tfrac{1}{2}\int_0^M (\tfrac{\partial r}{\partial t})^2 dm,\ U = \tfrac{1}{\gamma-1}\int_0^M \tfrac{p}{\rho} dm,\ V = -k\int_0^M \tfrac{m}{r} dm,\ \dot{M} \equiv \tfrac{dM(t)}{dt}, \end{gathered} \tag{2}$$

where T, U, V is kinetic, internal and potential (gravitational) gas energy, E is the explosion energy, $m = M(t)$ is law of shock wave motion by mass, $R(t) = r(M(t),t)$ is shock wave radius. Indices 1, 2 denote respectively the gas position before and after the surface of the strong discontinuity.
The boundary conditions on the discontinuity $r = R(t)$ in Eulerian coordinates are as follows:

$$[\rho(\dot{R} - \tfrac{\partial r}{\partial t})]_1^2 = 0, [p + \rho(\tfrac{\partial r}{\partial t} - \dot{R})^2]_1^2 = 0, [\tfrac{1}{2}(\tfrac{\partial r}{\partial t} - \dot{R})^2 + \tfrac{\gamma}{\gamma-1}\tfrac{p}{\rho}]_1^2 = 0, [\varphi]_1^2 \equiv \varphi_2 - \varphi_1. \tag{3}$$

If boundary conditions (3) are solved with respect to parameters of the gas behind the wave we get the following:

$$\begin{gathered} \rho_2 = \tfrac{\gamma+1}{\gamma-1}\rho_1[1 + \tfrac{1}{\gamma-1}\tfrac{2a_1^2}{(\dot{R}-(\frac{\partial r}{\partial t})_1)^2}]^{-1},\ a_1^2 = \tfrac{\gamma p_1}{\rho_1},\ p_2 = \tfrac{1}{\gamma+1}[p_1(1-\gamma) + 2\rho_1(\dot{R} - (\tfrac{\partial r}{\partial t})_1)^2], \\ \dot{R} - (\tfrac{\partial r}{\partial t})_2 = \tfrac{1}{\gamma+1}(\dot{R} - (\tfrac{\partial r}{\partial t})_1)[\gamma - 1 + \tfrac{2a_1^2}{(\dot{R}-(\frac{\partial r}{\partial t})_1)^2}]. \end{gathered} \tag{4}$$

Besides, the continuity of Euler's and Lagrange's variables ought to be taken into account.

$$[r]_1^2 = 0,\ [m]_1^2 = 0. \tag{5}$$

Initial conditions ($t = 0$, phone) determine the initial state of a gravitating gas sphere and are the exact solutions of (1) system.

Thus, the initial-boundary problem is considered in the domain Ω:

$$\Omega = \{t \in (0, t_*),\ m \in (0, M(t))\},$$

where $t = 0$ is the moment of explosion, t_* is the moment of time when the wave comes out on the surface of the body. Boundary conditions on the external unknown boundary $m = M(t)$ are like (4), (5) and in the center takes place

$$r(m,t) = 0,\ m = 0. \tag{6}$$

3 Exact solution before and approximate solution after shock wave. Let us discuss the problem of the central explosion at the $t = 0$ moment of a nonhomogeneous gas sphere (star) balanced in its own gravitation field.

Thus, the exact solution of the system of equations (1) that corresponds to the nonhomogeneous gas sphere balanced in its own gravitation field is taken as an initial condition. The gravitation constant k, the sphere center density ρ_c and the sphere radius a are taken as main units of dimension

$$\begin{gathered} \rho = 1 - r,\ m = 4\pi r^3(\tfrac{1}{3} - \tfrac{r}{4}),\ \tfrac{\partial r}{\partial t} = v = 0, \\ p = 4\pi[\tfrac{1}{6}(1 - r^2) - \tfrac{7}{36}(1 - r^3) + \tfrac{1-r^4}{16}]. \end{gathered} \tag{7}$$

Qualitative analysis of the system of equations (1) and boundary conditions (4) shows that the solution in the vicinity behind the shock wave can be sought in the form of the next singular decomposition

$$\begin{gathered} r = R_0(\tau) + \varepsilon H(m, \tau) + ...,\ R(\tau) = R_0(\tau) + \varepsilon R_1(\tau) + ..., \\ p = p_0(m, \tau) + \varepsilon p_1(m, \tau) + ...,\ \rho = \tfrac{\rho_0(m,\tau)}{\varepsilon} + \rho_1(m, \tau) + ..., \\ \tau = t/\sqrt{\varepsilon}, \varepsilon = \tfrac{\gamma-1}{\gamma+1} \ll 1. \end{gathered} \tag{8}$$

Substituting (8) singular decomposition into the system of equations (1), integral equation (2) and boundary conditions (4), we get a zero approximation to the solution of the problem

$$\begin{gathered} p_0(m, \tau) = R_0'^2(\tau)(1 - R_0(\tau)) + \tfrac{R_0''(\tau)(M_0(\tau)-m)}{4\pi R_0^2(\tau)},\ M_0(\tau) = 4\pi R_0^3(\tau)(\tfrac{1}{3} - \tfrac{R_0(\tau)}{4}), \\ \rho_0(m, \tau) = p_0^{\frac{1}{\gamma}}(m, \tau)[R_0'^2(T_0(m))]^{-\frac{1}{\gamma}}[1 + \tfrac{a_1^2(m)}{\gamma R_0'^2(T_0(m))}]^{-1}, \\ a_1^2(m) = \tfrac{4\pi\gamma}{1-r(m)}[\tfrac{1}{6}(1 - r^2(m)) - \tfrac{7}{36}(1 - r^3(m)) + \tfrac{(1-r^4(m))}{16}], \end{gathered} \tag{9}$$

where the function $r = r(m)$ is defined from the equation (7) and has the form

$$\begin{gathered} r(m) = \tfrac{1}{3} + \frac{\sqrt{2(A+B)+\frac{4}{9}} - \sqrt{\frac{8}{9} - 2(A+B) + \frac{16}{27\sqrt{2(A+B)+\frac{4}{9}}}}}{2}, \\ A(m) = \sqrt[3]{\tfrac{m}{9\pi}(1 + \sqrt{1 - \tfrac{3m}{\pi}})},\ B(m) = \sqrt[3]{\tfrac{m}{9\pi}(1 - \sqrt{1 - \tfrac{3m}{\pi}})}. \end{gathered} \tag{10}$$

The function $R_0(\tau)$ in (9) is the solution of the following Cauchy problem:

$$\pi[1 - R_0(\tau)]R_0'^2 R_0^3(\tau)[\tfrac{1}{3} - \tfrac{R_0(\tau)}{4}] = E_0,\ R_0(0) = 0 \tag{11}$$

and has the form

$$\int_0^{R_0} \sqrt{(1 - R_0)(4 - 3R_0)R_0^3} dR_0 = \sqrt{\frac{12E_0}{\pi}}\tau,\ \int x\sqrt{(1 - x)(4 - 3x)x}dx =$$

$$= \frac{2\{160\sqrt{x(1-x)(4-3x)}F[\sin^{-1}(\frac{2}{\sqrt{3x}})\ |_4^3] - 616\sqrt{x(1-x)(4-3x)}E[\sin^{-1}(\frac{2}{\sqrt{3x}})\ |_4^3] + y(x)\}}{2835\sqrt{x(1-x)(4-3x)}},$$

$$y(x) \equiv 1215x^5 - 3402x^4 + 2259x^3 - 84x^2 + 1244x - 1232,$$

where $F(\varphi, k), E(\varphi, k)$ are elliptic integrals of the first and second kind respectively.

At that $\tau \to 0_+$ asymptotics of Cauchy's problem solution (11) are calculated, as well as asymptotics at, where $\tau \to \tau_{*_-}$ ($\tau_* = 0,28287\sqrt{\frac{\pi}{12E_0}}$ is time of shock wave release to sphere surface)

$$R_0(\tau) \cong (\tfrac{75E_0}{4\pi})^{\frac{1}{5}}\tau^{\frac{2}{5}},\ \tau \to 0_+,\ R_0(\tau) \cong 1 - 3\sqrt[3]{\tfrac{E_0}{\pi}}(\tau_* - \tau)^{\frac{2}{3}},\ \tau \to \tau_{*_-}. \tag{12}$$

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