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ON THE CRITERION OF THE WELL-POSEDNESS FOR THE GENERAL BOUNDARY VALUE PROBLEMS FOR THE SYSTEMS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

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Abstract. The necessary and sufficient condition and effective sufficient conditions are presented for the well-posedness of the general linear boundary value problems for systems of ordinary differential equations.

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Let x_0 be the unique solution of the boundary value problem

$$\frac{dx}{dt} = P_0(t)x + q_0(t) \ \text{ for a.a. } \ t \in I, \ \ \ell_0(x) = c_0 \tag{1}$$

where $I = [a, b]$, $P_0 \in L(I; \mathbb{R}^{n\times n})$, $q_0 \in L(I; \mathbb{R}^n)$, $\ell_0 : C(I; \mathbb{R}^n) \to \mathbb{R}^n$ is a linear vector-functional, bounded with respect to the norm $\|.\|_c$, and $c_0 \in \mathbb{R}^n$.

Along with problem (1), consider the sequence of the problems

$$\frac{dx}{dt} = P_m(t)x + q_m(t) \ \text{ for a.a. } \ t \in I, \ \ \ell_m(x) = c_m \tag{1_m}$$

($m = 1, 2, \dots$), where $P_m \in L(I; \mathbb{R}^{n\times n})$, $q_m \in L(I; \mathbb{R}^n)$, $\ell_m : C(I; \mathbb{R}^{n\times n}) \to \mathbb{R}^n$ is a linear bounded vector-functional, and $c_m \in \mathbb{R}^n$.

We present the necessary and sufficient and effective sufficient conditions for problem (1_m) to have a unique solution x_m for any sufficiently large m and

$$\lim_{m\to+\infty} \|x_m - x_0\|_c = 0. \tag{2}$$

Similar question for the initial problem is investigated in [2, 5, 6] (see also references therein), where the sufficient and the necessary and sufficient conditions are obtained. In [1, 3, 4] the general linear boundary value problem (1) is investigated. The necessary and sufficient conditions have been proved in [1, 3] for the considered case.

Designations: $\mathbb{R} =]-\infty, +\infty[$, I_n is the identity $n \times n$-matrix, $O_{n\times n}$ and 0_n are, the zero $n \times n$-matrix and zero n-vector; $\|x\|_c = \max\{\|x(t)\| : t \in I\}$ is the norm of the vector-function $x : I \to \mathbb{R}^n$, $|||\ell|||$ is the norm of the linear bounded vector-functional ℓ.

Definition. We say that the sequence $(P_m, q_m; \ell_m)$ ($m = 1, 2, \dots$) belongs to the set $\mathcal{S}(P_0, q_0; \ell_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_m \in \mathbb{R}^n$ ($m = 1, 2, \dots$) satisfying condition $\lim\limits_{k\to+\infty} c_m = c_0$, problem (1_m) has a unique solution x_m for any sufficiently large m and condition (2) holds.

Theorem 1. *Let the conditions*

$$\lim_{m\to+\infty} \ell_m(x) = \ell_0(x) \ \ for \ \ x \in C(I;\mathbb{R}^n), \ \ \limsup_{m\to+\infty} |||\ell_m||| < \infty \tag{3}$$

hold. Then

$$\big((P_m, q_m; \ell_m)\big)_{m=1}^{\infty} \in \mathcal{S}(P_0, q_0; \ell_0) \tag{4}$$

if and only if there exist matrix-functions $H_m \in AC(I;\mathbb{R}^{n\times n})$ $(m = 0, 1, \dots)$ such that

$$\limsup_{m\to+\infty} \int\limits_a^b \|H_m'(t) + H_m(t)P_m(t)\|dt < \infty, \ \ \inf\big\{ |\det(H_0(t))| : \ t \in I\big\} > 0, \tag{5}$$

and uniformly on I the conditions hold

$$\lim_{m\to+\infty} H_m(t) = H_0(t), \tag{6}$$

$$\lim_{m\to+\infty} \int\limits_a^t H_m(t)P_m(t)dt = \int_a^t H_0(t)P_0(t)dt, \tag{7}$$

$$\lim_{m\to+\infty} \int\limits_a^t H_m(t)q_m(t)dt = \int_a^t H_0(t)q_0(t)dt. \tag{8}$$

Theorem 2. *Let (3) hold. Then inclusion (4) holds if and only if $\lim\limits_{m\to+\infty} X_m^{-1}(t) = X_0^{-1}(t)$, $\lim\limits_{m\to+\infty} \int\limits_a^t X_m^{-1}(\tau)q_m(\tau)\,d\tau = \int\limits_a^t X_0^{-1}(\tau)q_0(\tau)\,d\tau$ hold uniformly on I, where X_0 and X_m $(m = 1, 2, \dots)$ are the fundamental matrices of systems (1) and (1_m), respectively.*

Theorem 3. *Let $P_0^* \in L(I;\mathbb{R}^{n\times n})$, $q_0^* \in L(I;\mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, and a $\ell_0^* : C(I;\mathbb{R}^{n\times n}) \to \mathbb{R}^n$ be a linear bounded vector-functional such that the boundary value problem $\frac{dx}{dt} = P_0^*(t)x + q_0^*(t)$, $\ell_0^*(x) = c_0^*$ has a unique solution x_0^*. Let, moreover, there exist matrix- and vector-functions $H_m \in AC(I;\mathbb{R}^{n\times n})$ and $h_m \in AC(I;\mathbb{R}^n)$ $(m = 1, 2, \dots)$ such that $\inf\{|\det(H_m(t))| : t \in I\} > 0$ for every sufficiently large m, $\lim\limits_{m\to+\infty}(c_m + \ell_m^*(h_m)) = c_0^*$, $\lim\limits_{m\to+\infty} \ell_m^*(y) = \ell_0^*(y)$ $(y \in C(I;\mathbb{R}^n)$, $\limsup\limits_{m\to+\infty} |||\ell_m^*||| < \infty$, $\limsup\limits_{m\to+\infty} \int\limits_a^b \|P_m^*(t)\|dt < \infty$ and the conditions $\lim\limits_{m\to+\infty} \int\limits_a^t P_m^*(\tau)d\tau = \int\limits_a^t P_0^*(\tau)\,d\tau$, $\lim\limits_{m\to+\infty} \bigg(h_m(t) - h_m(a) + \int\limits_a^t \big(H_m(\tau)q_m(\tau)$ $-P_m^*(\tau)h_m(\tau)\big)d\tau\bigg) = \int\limits_a^t q_0^*(\tau)\,d\tau$ hold uniformly on I, where $P_m^*(t) \equiv (H_m'(t)+H_m(t)P_m(t))$ $\times H_m^{-1}(t)$, $\ell_m^*(y) \equiv \ell_m(H_m^{-1}y)$ $(m = 1, 2, \dots)$. Then problem (1_m) has the unique solution x_m for any sufficiently large m and $\lim\limits_{m\to+\infty} \|H_m\,x_m + h_m - x_0^*\|_c = 0$.*

Corollary 1. *Let conditions* (3), (5) *and* $\lim\limits_{m\to+\infty}(c_m-\varphi_m(a))=c_0$ *hold, and conditions* (6), (7) *and* $\lim\limits_{m\to+\infty}\int\limits_a^t\big(H_m(\tau)(q_m(\tau)-\varphi'_m(\tau))+P^*_m(\tau)\varphi_m(\tau)\big)d\tau=\int\limits_a^t H_0(\tau)q_0(\tau)d\tau$ *hold uniformly on* I, *where* $H_m\in AC(I;\mathbb{R}^{n\times n})$, $\varphi_m\in AC(I;\mathbb{R}^n)$ $(m=0,1,\dots)$. *Then problem* (1_m) *has a unique solution* x_m *for any sufficiently large* m *and* $\lim\limits_{m\to+\infty}\|x_m-\varphi_m-x_0\|_c=0$.

We give some effective sufficient conditions guaranteeing inclusion (4).

Theorem 4. *Let conditions* (3) *and* $\limsup\limits_{a\to+\infty}\int\limits_a^b\|P_m(t)\|\,dt<\infty$ *hold, and* $\lim\limits_{m\to+\infty}\int\limits_a^t P_m(\tau)\,d\tau=\int\limits_m^t P_0(\tau)\,d\tau$ *and* $\lim\limits_{m\to+\infty}\int\limits_a^t q_m(\tau)\,d\tau=\int\limits_a^t q_0(\tau)\,d\tau$ *uniformly on* I. *Then inclusion* (4) *holds.*

Corollary 2. *Let conditions* (3) *and* (5) *hold, and let conditions* (6), $\lim\limits_{m\to+\infty}\int\limits_a^t H_m(\tau)P_m(\tau)\,d\tau=\int\limits_a^t P^*(\tau)\,d\tau$ *and* $\lim\limits_{m\to+\infty}\int\limits_a^t H_m(\tau)q_m(\tau)\,d\tau=\int\limits_a^t q^*(\tau)\,d\tau$ *hold uniformly on* I, *where* $H_m\in AC(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$, $P^*\in L(I;\mathbb{R}^{n\times n})$, $q^*\in L(I;\mathbb{R}^n)$. *Let, moreover, the problem* $\frac{dx}{dt}=(P_0(t)-P^*(t))x+(q_0(t)-q^*(t))$, $\ell_0(x)=c_0$ *have a unique solution. Then* $\big((P_m,q_m;l_m)\big)_{m=1}^{\infty}\in\mathcal{S}(P_0-P^*,q_0-q^*;\ell_0)$.

Corollary 3. *Let* (3) *hold and let there exist a natural* μ *and matrix-functions* $B_j\in AC(I;\mathbb{R}^{n\times n})$ $(j=1,\dots,\mu-1)$ *such that* $\limsup\limits_{m\to+\infty}\int\limits_a^b\|H'_{m\,\mu-1}(t)+H_{m\,\mu-1}(t)P_m(t)\|dt<\infty$, *and uniformly on* I

$$\lim_{m\to+\infty}\left(H_{m\,j-1}(t)+\int\limits_a^t H_{m\,j-1}(\tau)P_m(\tau)d\tau\right)=I_n+B_j(t)-B_j(a)\ \ (j=1,\dots,\mu-1),$$

$$\lim_{m\to+\infty}\left(H_{m\,\mu-1}(t)+\int\limits_a^t H_{m\,\mu-1}(\tau)P_m(\tau)d\tau\right)=I_n+\int\limits_{t_0}^t P_0(\tau)\,d\tau,$$

$$\lim_{m\to+\infty}\int\limits_a^t H_{m\,\mu-1}(\tau)q_m(\tau)d\tau=\int\limits_a^t q_0(\tau)\,d\tau,$$

where $H_{mj}(t)\equiv-\left(H_{m\,j-1}(\tau)(t)+\int\limits_a^t H_{m\,j-1}(\tau)P_m(\tau)d\tau-B_j(t)+B_j(a)\right)H_{m\,j-1}(t)$ $(j=1,\dots,\mu-1;\ m=1,2,\dots)$, $H_{m0}(t)\equiv I_n$. *Then inclusion* (4) *holds.*

Corollary 3′. *Let conditions* (3) *and* (5) *hold, and let uniformly on* I *hold the conditions* $\lim\limits_{m\to+\infty}\int\limits_a^t P_m(\tau)\,d\tau=B(t)-B(a)$, $\lim\limits_{m\to+\infty}\int\limits_a^t H_m(\tau)P_m(\tau)\,d\tau=\int\limits_a^t P_0(\tau)\,d\tau$, $\lim\limits_{m\to+\infty}\int\limits_a^t H_m(\tau)\,q_m(\tau)\,d\tau=\int\limits_{t_0}^t q_0(\tau)\,d\tau$, *where* $B\in AC(I;\mathbb{R}^{n\times n})$, $H_0(t)\equiv O$, $H_m(t)\equiv I_n-\int\limits_a^t P_m(\tau)\,d\tau+B(t)-B(a)$ $(m=1,2,\dots)$. *Then inclusion* (4) *holds.*

Corollary 3′ is a case of Corollary 3, where $\mu=2$. It has the form if $B(t)\equiv\int\limits_a^t P_0(\tau)\,d\tau$.

Corollary 3″. *Let conditions* (3) *and* $\limsup\limits_{m\to+\infty}\int\limits_a^b \|I_n - H_m(t)\|\, P_m(t)dt < \infty$ *hold, and on* I *uniformly* $\lim\limits_{m\to+\infty} B_m(t) = O_{n\times n}$, $\lim\limits_{m\to+\infty}\int\limits_a^t B'_m(\tau)\left(\int\limits_a^\tau P_m(s)ds\right)d\tau = O_{n\times n}$, $\lim\limits_{m\to+\infty}\int\limits_a^t (I_n - B_m(\tau))\, q_m(\tau)d\tau = \int\limits_a^t q_0(\tau)d\tau$, *where* $B_m(t) \equiv \int\limits_a^t (P_m(\tau) - P_0(\tau))d\tau$ $(m = 1,2,\dots)$. *Then inclusion* (4) *holds.*

Corollary 4. *Let* (3) *hold. Then inclusion* (4) *holds if and only if there exist matrix-functions* $Q_m \in L(I;\mathbb{R}^{n\times n})$ $(m = 0,1,\dots)$ *such that* $\limsup\limits_{m\to+\infty}\int\limits_a^b \|P_m(t) - Q_m(t)\|\, dt < \infty$, *and the conditions* $\lim\limits_{m\to+\infty} Z_m^{-1}(t) = Z_0^{-1}(t)$, $\lim\limits_{m\to+\infty}\int\limits_a^t Z_m^{-1}(\tau)\, P_m(\tau)d\tau = \int\limits_a^t Z_0^{-1}(\tau)\, P_0(\tau)d\tau$, $\lim\limits_{m\to+\infty}\int\limits_a^t Z_m^{-1}(\tau)\, q_m(\tau)d\tau = \int\limits_a^t Z_0^{-1}(\tau)\, q_0(\tau)d\tau$ *hold uniformly on* I, *where* Z_m $(Z_m(a) = I_n)$ $(m = 1,2,\dots)$ *is a fundamental matrix of the system* $\frac{dx}{dt} = Q_m(t)x$.

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