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ON THE CRITERION OF THE CONVERGENCE OF DIFFERENCE SCHEMES FOR LINEAR GENERAL BOUNDARY PROBLEMS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The necessary and sufficient condition and effective sufficient conditions are presented for the convergence difference schemes corresponding to the general linear boundary value problems for systems of ordinary differential equations.

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Let the vector-function $x_0 : I \to \mathbb{R}^n$ be the unique solution of problem

$$\frac{dx}{dt} = P(t)x + q(t) \ \text{ for a.a. } \ t \in I, \ \ \ell(x) = c_0 \tag{1}$$

$I = [a,b]$, $P \in L(I;\mathbb{R}^{n\times n})$, $q \in L(I;\mathbb{R}^n)$, $\ell : C(I;\mathbb{R}^n) \to \mathbb{R}^n$ is a linear vector-functional, bounded with respect to the norm $\|.\|_c$, and $c_0 \in \mathbb{R}^n$.

Along with the problem we consider the difference scheme

$$\Delta y(k-1) = \frac{1}{m}\big(G_{1m}(k)\, y(k) + G_{2m}(k-1)\, y(k-1) + g_{1m}(k) + g_{2m}(k-1)\big)$$
$$(k = 1,\dots,m), \ \ \mathcal{L}_m(y) = \gamma_m \tag{1_m}$$

$(m = 1,2,\dots)$, where $G_{jm} \in \mathrm{E}(N_m;\mathbb{R}^{n\times n})$, $g_{jm} \in \mathrm{E}(N_m;\mathbb{R}^n)$ $(j = 1,2)$ and $\mathcal{L}_m : \mathrm{E}(J;\mathbb{R}^{n\times m}) \to \mathbb{R}^n$ is a given linear bounded vector-functional. In addition, assume $G_{1m}(0) = G_{2m}(m) = O_{n\times n}$ and $g_{1m}(0) = g_{2m}(m) = 0_n$ $(m \in \mathbb{N})$.

We present the effective necessary and sufficient (moreover, the effective sufficient) conditions for the convergence of the solution of difference scheme (1_m) to x_0. They are proved in [2]. The analogous problem is investigated in [1] for the initial one.

Designations. $\mathbb{R} =]-\infty,+\infty[$, $\mathbb{N} = \{1,2,\dots\}$, $\mathbb{N}_m = \{1,\dots,m\}$, $\widetilde{\mathbb{N}}_m = \{0,1,\dots,m\}$; I_n is the identity $n\times n$-matrix, $O_{n\times n}$ and 0_n are, the zero $n\times n$-matrix and zero n-vector, respectively; $\|x\|_c = \max\{\|x(t)\| : t \in I\}$ is the norm of the vector-function $x : I \to \mathbb{R}^n$, $|||\ell|||$ is the norm of the linear bounded vector-functional ℓ. If $J \subset \mathbb{N}$, then $\mathrm{E}(J;\mathbb{R}^{n\times m})$ is the space of all bounded matrix-functions $Y : J \to \mathbb{R}^{n\times m}$ with the norm $\|Y\|_J = \max\{\|Y(k)\| : k \in J\}$.

Let $\Delta Y(i-1) \equiv Y(i) - Y(i-1)$ for $Y \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times m})$. Further, $\tau_m = (b-a)/m$, $\tau_{0m} = a$, $\tau_{km} = a + k\tau_m$ and $I_{km} =]\tau_{k-1\,m}, \tau_{km}[$ $(k \in \mathbb{N}_m;\ m \in \mathbb{N})$. Let ν_m be function defined by

$\nu_m(t) \equiv [(t-a)(b-a)^{-1}m]$ $(m \in \mathbb{N})$, where $[T]$ stands for the integer part of T. Obviously, $\nu_m(\tau_{km}) = k$ $(k \in \widetilde{\mathbb{N}}_m;\ m \in \mathbb{N})$. $p_m : \mathrm{BV}([a,b];\mathbb{R}^n) \to \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ and $q_m : \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n) \to \mathrm{BV}([a,b];\mathbb{R}^n)$ $(m \in \mathbb{N})$ are operators defined, respectively, by $p_m(x)(k) = x(\tau_{km})$ and $q_m(y)(t) \equiv \begin{cases} y(k) \ \text{ if } \ t = \tau_{km} \ \text{ for some } \ k \in \widetilde{\mathbb{N}}_m, \\ y(k) - \frac{1}{m}G_{1m}(k)y(k) - \frac{1}{m}g_{1m}(k) \ \text{ if } \ t \in]\tau_{k-1\,m}, \tau_{k\,m}[\ \text{ for some } \ k \in \widetilde{\mathbb{N}}_m. \end{cases}$

Definition. The inclusion

$$\big((G_{1m}, G_{2m}, g_{1m}, g_{2m}; \mathcal{L}_m)\big)_{m=1}^{+\infty} \in \mathcal{CS}(P, q; \ell) \tag{2}$$

means that for every $c_0 \in \mathbb{R}^n$ and the sequence $\gamma_m \in \mathbb{R}^n$ $(m \in \mathbb{N})$, satisfying the condition $\lim\limits_{m\to+\infty} \gamma_m = c_0$, the difference problem (1_m) has a unique solution $y_m \in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ for any sufficiently large m and $\lim\limits_{m\to+\infty} \|y_m - p_m(x_0)\|_{\widetilde{N}_m} = 0$.

Theorem 1. *Let*

$$\lim_{m\to+\infty} \mathcal{L}_m(p_m(x)) = \ell(x) \quad \text{for } \ x \in \mathrm{BV}(I;\mathbb{R}^n), \quad \text{and} \ \limsup_{m\to+\infty} |||\mathcal{L}_m||| < +\infty. \tag{3}$$

Then inclusion (2) *holds if and only if there exist matrix-functions* $H \in AC(I;\mathbb{R}^{n\times n})$ *and* $H_{1m},\ H_{2m} \in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ $(m \in \mathbb{N})$ *such that* $\inf\{|\det(H(t))| : t \in I\} > 0$,

$$\limsup_{m\to+\infty} \sum_{k=1}^{m} \big(\big\|H_{2m}(k) - Q_{1m}(k)\big\| + \big\|H_{1m}(k) - Q_{2m}(k))\big\|\big) < +\infty, \tag{4}$$

$$\lim_{m\to+\infty} \max_{k\in\widetilde{\mathbb{N}}_m} \{\|H_{jm}(k) - H(\tau_{km})\|\} = 0 \ \ (j = 1, 2), \tag{5}$$

where $Q_{j+1\,m}(k) \equiv H_{jm}(k-j) - \frac{1}{m}H_{1m}(k)\,G_{j+1\,m}(k-j)$ $(j = 0, 1)$, *and uniformly on* I

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{k=1}^{\nu_m(t)} \sum_{j=0}^{1} H_{1m}(k)\,G_{j+1m}(k-j) = \int_a^t H(\tau)P(\tau)d\tau, \tag{6}$$

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{k=1}^{\nu_m(t)} \sum_{j=0}^{1} H_{1m}(k)\,g_{j+1m}(k-j) = \int_a^t H(\tau)q(\tau)d\tau. \tag{7}$$

Remark. The limits equalities (6) and (7) are fulfilled uniformly on I if, respectively, $\lim\limits_{m\to+\infty} \max\limits_{i\in\mathbb{N}_m} \Big\{\Big|\frac{1}{m} \sum\limits_{k=1}^{i} \sum\limits_{j=0}^{1} H_{1m}(k)\,G_{j+1\,m}(k-j) - \int\limits_a^{\tau_{im}} H(\tau)P(\tau)d\tau\Big|\Big\} = O_{n\times n}$,
$\lim\limits_{m\to+\infty} \max\limits_{i\in\mathbb{N}_m} \Big\{\Big|\frac{1}{m} \sum\limits_{k=1}^{i} \sum\limits_{j=0}^{1} H_{1m}(k)\,g_{j+1\,m}(k-j) - \int\limits_a^{\tau_{im}} H(\tau)q(\tau)d\tau\Big|\Big\} = 0_n$.

Let X and Y_m be the normal fundamental matrices of systems (1) and (1_m) $(m \in \mathbb{N})$.

Theorem 2. *Let conditions* (3) *and* $\det\Big(I_n + (-1)^j \frac{1}{m} G_{jm}(k)\Big) \neq 0$ $(j = 1, 2;\ k \in \mathbb{N}_m$ $m \in \mathbb{N})$ *hold. Then inclusion* (2) *holds if and only if* $\lim\limits_{m\to+\infty} \max\limits_{k\in\widetilde{\mathbb{N}}_m} \{\|Y_m^{-1}(k) - X^{-1}(\tau_{km})\|\}$ $= 0$ *and* $\lim\limits_{m\to+\infty} \max\limits_{i\in\mathbb{N}_m} \Big\{\Big|\frac{1}{m} \sum\limits_{k=1}^{i} \sum\limits_{j=0}^{1} Y_m^{-1}(k)\,g_{j+1\,m}(k-j) - \int\limits_a^{\tau_{im}} X^{-1}(\tau)q(\tau)d\tau\Big|\Big\} = 0_n$.

If P satisfied the Lappo–Danilevskiĭ condition then $X(t) \equiv \exp\Big(\int\limits_s^t P(\tau)\,d\tau\Big)$. Further, $Y_m(k) \equiv \prod\limits_{i=k}^{1}\Big(I_n - \frac{1}{m}G_{1m}(i)\Big)^{-1}\Big(I_n + \frac{1}{m}G_{2m}(i-1)\Big)$ $(m \in \mathbb{N})$. In Theorem 1, condition (4) automatically holds because Y_m is the fundamental matrix of system (1_m) $(m \in \mathbb{N})$.

Now we give a method of constructing of discrete real matrix-and vector-functions, respectively, G_{jm} and g_{jm} $(j = 1,2;\ m \in \mathbb{N})$ for which the conditions of Theorem hold.

We use the inductive method. Let $\mathcal{E}_m \in E(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ and let $\xi_m \in E(\widetilde{\mathbb{N}}_m;\mathbb{R}^n)$ $(m \in \mathbb{N})$ be such that $\lim\limits_{m\to+\infty}\|\mathcal{E}_m\|_{\widetilde{\mathbb{N}}_m} = 0$ and $\lim\limits_{m\to+\infty} m\|\xi_m\|_{\widetilde{\mathbb{N}}_m} = 0$. Let $P_{lm} = X(\tau_{lm}) + \mathcal{E}_m(l)$, and let $G_{1m}(1)$ and $G_{2m}(0)$ be such that $Y_m(1) = P_{1m}$ $(l \in \widetilde{\mathbb{N}}_m\ m \in \mathbb{N})$. It is evident that $\big(I_n - m^{-1}G_{1m}(1)\big)^{-1}\big(I_n + m^{01}G_{2m}(0)\big) = P_{1m}$. So, $G_{1m}(1)$ and $G_{2m}(0)$ are arbitrary matrices such that $G_{1m}(1) = m\big(I_n - P_{1m}^{-1}\big) - G_{2m}(0)\,P_{1m}^{-1}$. Let $G_{1m}(k)$, $G_{2m}(k-1)$ and $Y_m(k)$ $(k = 1,\dots,l-1)$ be constructed. For the construction $G_{1m}(l)$ and $G_{2m}(l-1)$ we use the equalities $Y_m(l) = P_{lm}$ and $Y_m(l) = \big(I_n - m^{-1}G_{1m}(l)\big)^{-1}\big(I_n + m^{-1}G_{2m}(l-1)\big)Y_m(l-1)$. As above, we obtain the relation $G_{1m}(l) = m\big(I_n - P_{l-1\,m}P_{lm}^{-1}\big) - G_{2m}(l-1)\,P_{l-1\,m}\,P_{lm}^{-1}$. So, $G_{1m}(l)$ and $G_{2m}(l-1)$ will be an arbitrary matrix satisfying the last equality.

Let us now construct the discrete vector-functions g_{jm} $(j = 1,2;\ m \in \mathbb{N})$. As $g_{1m}(l)$ and $g_{2m}(l-1)$ we choose the vectors such that $m^{-1}Y_m^{-1}(l)(g_{1m}(1) + g_{2m}(l-1)) = q_{lm}$ $(l \in \mathbb{N}_m)$, where $q_{lm} \equiv \xi_m(l) + \int\limits_a^{\tau_{lm}} X^{-1}(\tau)q(\tau)d\tau$ $(l \in \mathbb{N}_m)$. Therefore, we have the following equalities $g_{1m}(l) + g_{2m}(l-1) = mY_m(l)q_{lm}$ $(l \in \mathbb{N}_m)$ for the definition of g_{jm} $(j = 1,2;\ m \in \mathbb{N})$.

We realize constructed above discrete matrix-and vector-functions for the example.

Example. Let $X(t) \equiv \exp\Big(\int\limits_a^t P(\tau)d\tau\Big)$ be the fundamental matrix of system (1) and let $\mathcal{E}_m(k) \equiv O_{n\times n}$ and $\xi_m(k) \equiv 0_n$ for $m \in \mathbb{N}$. Then $P_{lm} = \exp\Big(\int\limits_a^{\tau_{lm}} P(\tau)d\tau\Big)$ $(l \in \widetilde{\mathbb{N}}_m,\ m \in \mathbb{N})$. If we choose $G_{2m}(l-1) = P_{lm}P_{l-1\,m}^{-1} = \exp\Big(\int\limits_{\tau_{l-1\,m}}^{\tau_{lm}} P(\tau)d\tau\Big)$ $(l \in \mathbb{N}_m,\ m \in \mathbb{N})$, then $G_{1m}(l) \equiv (m-1)I_n - m\exp\Big(-\int\limits_{\tau_{l-1\,m}}^{\tau_{lm}} P(\tau)d\tau\Big)$. For the definitions of g_{jm} $(j = 1,2)$ we have the relations $g_{1m}(l) + g_{2m}(l-1) \equiv m\int\limits_a^{\tau_{lm}} C(\tau_{lm},\tau)q(\tau)d\tau$ $(m \in \mathbb{N})$, where $C(t,\tau)$ is the Cauchy matrix of system (1). In particular, we can take $g_{1m}(l) \equiv \alpha\, m\int\limits_a^{\tau_{lm}} C(\tau_{lm},\tau)q(\tau)d\tau$ and $g_{2m}(l-1) \equiv (1-\alpha)\, m\int\limits_a^{\tau_{lm}} C(\tau_{lm},\tau)q(\tau)d\tau$ $(m \in \mathbb{N})$, where α is some number.

Theorem 3. *Let conditions* (3), $\limsup\limits_{m\to+\infty}\sum\limits_{k=1}^{m}\Big(\frac{1}{m}(\|G_{1m}(k)\| + \|G_{2m}(k-1)\|)\Big) < \infty$ *hold, and* $\lim\limits_{m\to+\infty}\frac{1}{m}\sum\limits_{k=1}^{\nu_m(t)}\sum\limits_{j=0}^{1} G_{j+1\,m}(k-j) = \int\limits_a^t P(\tau)d\tau$, $\lim\limits_{m\to+\infty}\frac{1}{m}\sum\limits_{k=1}^{\nu_m(t)}\sum\limits_{j=0}^{1} g_{j+1\,m}(k-j) = \int\limits_a^t q(\tau)d\tau$

hold uniformly on I. Then inclusion (2) *holds.*

Proposition. Let conditions (3), (4), (5), $\lim\limits_{m\to+\infty}\frac{1}{m}\max\limits_{k\in\widetilde{\mathbb{N}}_m}\{\|G_{jm}(k)\|+\|g_{jm}(k)\|\}=0$ ($j=1,2$) hold, and (6), (7) hold uniformly on I, where $H\in AC(I;\mathbb{R}^{n\times n})$, H_{1m}, $H_{2m}\in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$ ($m\in\mathbb{N}$). Let, moreover, either $\limsup\limits_{m\to+\infty}\Big(\frac{1}{m}\sum\limits_{k=0}^{m}(\|G_{jm}(k)\|+\|g_{jm}(k)\|)\Big)<\infty$ ($j=1,2$) or $\limsup\limits_{m\to+\infty}\sum\limits_{k=0}^{m}\sum\limits_{j=0}^{1}\|H_{1m}(k)-H_{2m}(k-j)\|<\infty$. Then inclusion (2) holds.

Corollary. Let condition (3) hold and let there exist a natural μ and matrix-functions $B_{jl}\in \mathrm{E}(\widetilde{\mathbb{N}}_m;\mathbb{R}^{n\times n})$, $B_{jl}(a)=O_{n\times n}$ ($j=1,2$; $l=0,\dots,\mu-1$) such that

$\limsup\limits_{m\to+\infty}\sum\limits_{k=1}^{m}\big(\big\|H_{2m}(k)-Q_{1m\mu}(k)\big\|+\big\|H_{1m}(k)-Q_{2m\mu}(k))\big\|\big)<\infty$,

$\lim\limits_{m\to+\infty}\max\limits_{k\in\widetilde{\mathbb{N}}_m}\{\|H_{jm\mu}(k)-I_n\|\}=0$ ($j=1,2$), where $Q_{j+1\,m\mu}(k)\equiv H_{jm\mu}(k-j)$

$-\frac{1}{m}H_{1m\mu}(k)\,G_{j+1\,m\mu}(k-j)$ ($j=0,1$), and $\lim\limits_{m\to+\infty}\frac{1}{m}\sum\limits_{k=1}^{\nu_m(t)}\sum\limits_{j=0}^{1}G_{j+1\,m\mu}(k-j)=\int\limits_a^t P(\tau)d\tau$,

$\lim\limits_{m\to+\infty}\frac{1}{m}\sum\limits_{k=1}^{\nu_m(t)}\sum\limits_{j=0}^{1}g_{j+1\,m\mu}(k-j)=\int\limits_a^t q(\tau)d\tau$ hold uniformly on I, where $H_{1m0}(k)=I_n$,
$H_{1m\,l+1}(k)\equiv\big(\frac{1}{m}H_{1ml}(k)\,G_{1m}(k)+\mathcal{Q}_1(H_{1ml},G_{1m},G_{2m})(k)+B_{1\,l+1}(k)\big)H_{1ml}(k)$,
$H_{2m0}(k)\equiv I_n$, $H_{2m\,l+1}(k)\equiv\big(\mathcal{Q}_2(H_{1ml},G_{1m},G_{2m})(k)+B_{2\,l+1}(k)\big)H_{2ml}(k)$,
$G_{1m\,l+1}(k)\equiv H_{1ml}(k)G_{1m}(k)$, $G_{2m\,l+1}(k)\equiv H_{1ml}(k+1)G_{2m}(k)$,
$g_{1m\,l+1}(k)\equiv H_{ml}(k)g_{1m}(k)$, $g_{2m\,l+1}(k)\equiv H_{ml}(k+1)g_{2m}(k)$,
$\mathcal{Q}_j(H_{1ml},G_{1m},G_{2m})(k)\equiv 2I_n-H_{jml}(k)-\frac{1}{m}\sum\limits_{i=1}^{k}H_{1ml}(i)\,(G_{1m}(i)+G_{2m}(i-1))$
($j=1,2$; $l=0,\dots,\mu-1$; $m=1,2,\dots$). Then inclusion (2) holds.

Everywhere, without loss of generality, we can assume that $H(t)\equiv I_n$.

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