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REPRESENTATION FUNCTIONS OF BINARY QUADRATIC FORMS BELONGING TO MULTI-CLASS GENERA

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Abstract. In this paper we show that the problem of obtaining formulas for the representation functions in case of binary quadratic forms belonging to multi-class genera can be reduced to the case of one-class genera.

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Let $r(n; f)$ denote a number of representations of a positive integer n by a positive definite binary quadratic form $f = ax^2 + bxy + cy^2$.

If the genus of the quadratic form f contains one class, then according to Newman Hall's theorems [[1], p. 592–595], the problem of obtaining "exact" formulas for $r(n; f)$ is solved completely. The values of functions $r(n; f)$ can be calculated by the formulas for the average number of representations of a positive integer by a genus that contains the form f (Vepkhvadze T. [2]).

Using the Gauss theory of genera, Kaplan and Williams ([3], p. 87, 88) computed the representation functions $r(n; f)$ for some n in the case of binary forms, belonging to multi-class genera. In [4] we proved three theorems which show that the problem of computing the representation function in case of multi-class genera can be reduced to the case of one-class genera.

Theorem 1 ([4], p. 277)**.** *Let $f = ax^2 + by^2$ be a primitive positive binary form, $ab \equiv 3 \pmod 4$. Then*

$$r(2^\alpha m; f) = r(2^{\alpha-2}m; f_1),$$

where $\alpha \geq 2$, $2 \nmid m$, $f_1 = ax^2 + axy + \frac{a+b}{4}y^2$.

Theorem 2 ([4], p. 277)**.** *Let $f = ax^2 + by^2$ be a primitive positive binary form, $\Delta = ab \equiv 0 \pmod 4$. Then*

$$r(2^\alpha m; f) = r(2^{\alpha-2}m; f_1),$$

where $2|\alpha$, $\alpha \geq 2$, $2 \nmid m$, $f_1 = ax^2 + \frac{b}{4}y^2$.

Theorem 3 ([4], p. 278)**.** *Let $f = ax^2 + axy + by^2$ be a primitive positive binary form with odd discriminant $d = a^2 - 4ab$, $\Delta = -d$, $9|2a + b$. Then*

$$r(3^\beta u; f) = r(3^{\beta-2}u; f_1),$$

where $\beta \geq 2$, $3 \nmid u$, $f_1 = ax^2 + axy + \frac{2a+b}{9}y^2$.

Theorem 3 can be extended as follows:

Theorem 4. *Let $f = ax^2 + bxy + cy^2$ be a primitive binary quadratic form with the odd discriminant $d = b^2 - 4ac$ and let p be a prime number with $p^2|b^2 - 4ac$, $p^2|a\delta^2 + b\delta + c$ for some integer δ. If $2|\alpha$, $\alpha \geq 2$ and $p \nmid m$, then*

$$r(p^\alpha m; f) = r(p^{\alpha-2}m; f_1),$$

where

$$f_1 = ax^2 + \frac{2a(p-\delta) - b}{p} xy + \frac{a(p-\delta)^2 - b(p-\delta) + c}{p^2} y^2.$$

Applying the Theorem 4 to the discriminant $d = -175$, we obtain the following

Corollary 1. *Let $f^{(1)} = x^2 + xy + 44y^2$, $f^{(2)} = 4x^2 + xy + 11y^2$, $f^{(3)} = 4x^2 - xy + 11y^2$, $n = 2^\alpha 5^\beta 7^\gamma u$ with $(u, 70) = 1$. Then*

$$\begin{aligned} r(n; f^{(1)}) =& r(n; f^{(2)}) = r(n; f^{(3)}) \\ =& (\alpha+1)\Big(1 + \Big(\frac{u}{7}\Big)\Big) \sum_{\nu|u} \Big(\frac{-7}{\nu}\Big) \quad \textit{if} \quad 2|\beta, \quad \beta \geq 2; \\ =& 0 \quad \textit{if} \quad 2 \nmid \beta. \end{aligned}$$

Now we consider the set of binary quadratic forms with the discriminant $d = b^2 - 4ac = -204$. This set splits into two genera, each consisting of three classes with reduced forms, respectively,

$$f^{(4)} = x^2 + 51y^2, \quad f^{(5)} = 4x^2 + 2xy + 13y^2, \quad f^{(6)} = 4x^2 - 2xy + 13y^2$$

and

$$f^{(7)} = 3x^2 + 17y^2, \quad f^{(8)} = 5x^2 + 4xy + 11y^2, \quad f^{(9)} = 5x^2 - 4xy + 11y^2.$$

Applying Theorem 1 to this discriminant, we obtain the following

Corollary 2. *Let $n = 2^\alpha 3^\beta 17^\gamma u$ with $(u, 102) = 1$. Then*

$$\begin{aligned} r(n; f^{(4)}) =& r(n; f^{(5)}) = r(n; f^{(6)}) \\ =& \frac{1}{2}\Big(1 + (-1)^{\beta+\gamma}\Big(\frac{u}{3}\Big)\Big)\Big(1 + (-1)^{\beta+\gamma}\Big(\frac{u}{17}\Big)\Big) \sum_{\nu|u} \Big(\frac{-51}{\nu}\Big) \quad \textit{if} \quad 2|\alpha, \quad \alpha \geq 2; \\ =& 0 \quad \textit{if} \quad 2 \nmid \alpha. \end{aligned}$$

$$\begin{aligned} r(n; f^{(7)}) =& r(n; f^{(8)}) = r(n; f^{(9)}) \\ ] =& \frac{1}{2}\Big(1 + (-1)^{\beta+\gamma+1}\Big(\frac{u}{3}\Big)\Big)\Big(1 + (-1)^{\beta+\gamma+1}\Big(\frac{u}{17}\Big)\Big) \sum_{\nu|u} \Big(\frac{-51}{\nu}\Big) \quad \textit{if} \quad 2|\alpha, \quad \alpha \geq 2; \\ =& 0 \quad \textit{if} \quad 2 \nmid \alpha. \end{aligned}$$

If n is an odd number, then in case of binary forms with the discriminant $d=-204$, we have

$$\begin{aligned} r(n;f^{(5)}) =& r(n;f^{(6)}) = \sum_{4x^2+2xy+13y^2=n} 1 \\ =& \frac{1}{2} \sum_{x^2+xy+13y^2=n} (1+(-1)^x) \\ =& \frac{1}{2} \sum_{x^2+xy+13y^2=n} + \frac{1}{2} \sum_{x^2+xy+13y^2=n} (-1)^x; \end{aligned}$$

$$\begin{aligned} r(n;f^{(8)}) =& r(n;f^{(9)}) \\ =& \sum_{5x^2+4xy+11y^2=n} 1 = \sum_{12x^2+6xy+5y^2=n} 1 \\ =& \frac{1}{2} \sum_{3x^2+3xy+5y^2=n} (1+(-1)^x) \\ =& \frac{1}{2} \sum_{3x^2+3xy+5y^2=n} 1 + \frac{1}{2} \sum_{3x^2+3xy+5y^2=n} (-1)^x. \end{aligned}$$

As the forms $x^2+xy+13y^2$ and $3x^2+3xy+5y^2$ belong respectively to one class genera, then the corresponding representation functions can be calculated by the formulas for an average number of representations of a positive integer by the genus, consisting these forms.

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