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ON JENSEN'S FUNCTIONAL EQUATION

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Abstract. Our aim is to give a martingale characterization to the general measurable solution of the Jensen functional equation. We show that the function $f = (f(x), x \in R)$ is a measurable solution of Jensen's functional equation if and only if the process $f(x + W_t)$ is a martingale for all $x \in R$, where W is a standard Brownian Motion.

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1 Introduction. Let $W = (W_t, t \geq 0)$ be a standard Brownian Motion, defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $F = (\mathcal{F}_t, t \geq 0)$ be a filtration generated by the Brownian Motion $\mathcal{F}_t^W = \sigma(W_s, s \leq t)$.

We consider the Jensen functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}, \quad x, y \in R. \tag{1}$$

It is well known that the general measurable solution of (1) is linear (see e.g. [1]). In the following theorem, which is our main result, we give an equivalent characterization of the general measurable solution of (1) in terms of martingale transformation of the Brownian motion.

Theorem. *The function $f = (f(x), x \in R)$ is a measurable solution of equation* (1) *if and only if the process $f(x + W_t)$ is a martingale for all $x \in R$.*

2 The proof of the main result. We shall often use the equality

$$f\left(\frac{x}{2}\right) = \frac{f(x) + f(0)}{2}, \tag{2}$$

which follows from (1) taking $y = 0$. Before we start off with the proof of the main result let us bring the next

Lemma. *Let ξ be a random variable having normal distribution with $E\xi = 0$. If $f(x)$ is a measurable solution of Jensen's functional equation* (1) *then $f(\xi)$ is also normally distributed and*

$$Ef(\xi) = f(0).$$

Proof. To show that $f(\xi)$ is normally distributed, as in [3], we use the Bernstein theorem. Let ξ and η be independent random variables, having normal distribution with zero mean and equal variances. Let

$$X = f(\xi) \quad \text{and} \quad Y = f(\eta).$$

It follows from (1) and (2) that

$$\frac{X+Y}{2} = \frac{f(\xi)+f(\eta)}{2} = f\left(\frac{\xi+\eta}{2}\right) = \frac{f(\xi+\eta)+f(0)}{2}$$

and hence

$$X + Y = f(\xi+\eta) + f(0). \tag{3}$$

Putting $x = \xi - \eta$ and $y = \eta$ in (1)

$$\frac{f(\xi-\eta)+f(\eta)}{2} = f\left(\frac{\xi}{2}\right) = \frac{f(\xi)+f(0)}{2},$$

which implies that

$$X - Y = f(\xi-\eta) - f(0). \tag{4}$$

Since $\xi + \eta$ and $\xi - \eta$ are independent random variables, it follows from (3) and (4) that the random variables $X + Y$ and $X - Y$ are also independent. Therefore, by Bernstein's theorem [2] we conclude that $f(\xi)$ is normally distributed. This implies that

$$E\,|f(\xi)| < \infty.$$

Now in equation (1) we can put $x = \xi$ and $y = -\xi$, which gives us

$$f(\xi) + f(-\xi) = 2f(0).$$

Taking expectations in the last equation on both sides and considering the fact that ξ has symmetrical distribution we get

$$Ef(\xi) = f(0).$$

$\square$

The proof of the Theorem. Assume that the function $f = (f(x), x \in R)$ is a measurable solution of equation (1). We need to show that $f(W_t)$ is a martingale. By the Lemma $f(W_t)$ is integrable for any $t \geq 0$ and we have to show the martingale equality

$$E\left(f(W_t)\,|\mathcal{F}_s\right) = f(W_s) \quad \text{a.s.} \tag{5}$$

Taking $x = W_t - W_s$ and $y = W_s$ in equation (1) and using also equation (2) for $x = W_t$ we have

$$f\left(\frac{W_t}{2}\right) = \frac{f(W_t - W_s) + f(W_s)}{2} = \frac{f(W_t)+f(0)}{2},$$

from where we obtain

$$f(W_t) = f(W_t - W_s) + f(W_s) - f(0).$$

If we take conditional expectations in the last equation we get

$$E(f(W_t)|\mathcal{F}_s) = E(f(W_t - W_s) + f(W_s) - f(0)|\mathcal{F}_s) \quad a.s.$$

As long as $W_t - W_s$ is independent of $\mathcal{F}_s$ filtration and W_s is $\mathcal{F}_s$ measurable , by the properties of conditional expectation we will have

$$E(f(W_t)|\mathcal{F}_s) = Ef(W_t - W_s) + f(W_s) - f(0) \quad a.s. \tag{6}$$

Since by the Lemma $Ef(W_t - W_s) = f(0)$, the martingale equality (5) follows from equation (6). Now it is easy to show that $f(x + W_t)$ is also a martingale. From (1) and (2) we will have

$$f\left(\frac{x + W_t}{2}\right) = \frac{f(x) + f(W_t)}{2} = \frac{f(x + W_t) + f(0)}{2}, \tag{7}$$

which implies that from (7) we have

$$f(x + W_t) = f(x) + f(W_t) - f(0),$$

from where it is evident that $f(x + W_t)$ is also a martingale.

Now, if we assume that $f(x + W_t)$ is a martingale for every $x \in R$, then Theorem A1 from [4] implies that the function f is of the form

$$f(x) = ax + b, \quad a, b \in R, \tag{8}$$

which satisfies equation (1). $\square$

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