Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 34, 2020

ABOUT SOME METHODS OF ANALYTIC REPRESENTATION AND CLASSIFICATION OF A WIDE SET OF GEOMETRIC FIGURES WITH "COMPLEX" CONFIGURATION

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Abstract. We will present 2 different analytical representations of only one general idea - this is the representation of complex movements using the superposition of certain elementary displacements! Despite of the analytical and structural similarity of these representations, they describe fundamentally different geometric figures (in statics) and trajectories of motion (in dynamics). We show some geometric properties of GRT,DRT,GML and DML - surfaces.

Keywords and phrases: Analytic representation, Möbius-Listing's surfaces, Jacobian, One-to one transformation, Regular points, Self-cross points.

AMS subject classification (2010): 53A05, 51B10, 57M25.

• **Notations and abbreviations.** In this article we use the following notations:
- X, Y, Z, t - is the ordinary notation for space and time coordinates;
- τ, ψ, θ - are local coordinates in parallelogram, where:

$$\begin{aligned} &1.\ \tau \in [-\tau^*, \tau^*], \text{where } 0 < \tau^*;\\ &2.\ \psi \in [0, 2\pi];\\ &3.\ \theta \in [0, 2\pi h], \quad \text{where} \quad h \in \mathbf{R}\ (Real). \end{aligned} \tag{1}$$

Further we will use the notation accepted in earlier works [1 - 3]. As a consequence of this, each figure is represented by analytical representations (2) and (6), which, geometrically meaning in static (i.e. $t = t_0$) is, correspondence the points of the parallelogram(1) (τ, ψ, θ) to the points of this or that figure (X, Y, Z), but in dynamic these are traces of complicated movements of these figures.

• **Generalized twisting and rotated bodies in dynamics** - GTR^n_m (sometimes called **"Surfaces of revolution"**) are defined by the parametric representations:

$$\begin{aligned} X(\tau, \psi, \theta, t) &= T_1(t) + [R(\psi, \theta, t) + p(\tau, \psi, \theta, t)\cos(\psi + \mu g(\theta))]\cos(\theta + M(t)),\\ Y(\tau, \psi, \theta, t) &= T_2(t) + [R(\psi, \theta, t) + p(\tau, \psi, \theta, t)\cos(\psi + \mu g(\theta))]\sin(\theta + M(t)),\\ Z(\tau, \psi, \theta, t) &= T_3(t) + Q(\theta, t) + p(\tau, \psi, \theta, t)\sin(\psi + \mu g(\theta)). \end{aligned} \tag{2}$$

For simplicity, we introduce the following abbreviations $R \equiv R(\psi, \theta, t)$, $p \equiv p(\tau, \psi, \theta, t)$, $R_\psi \equiv \frac{\partial R}{\partial \psi}$, $R_\theta \equiv \frac{\partial R}{\partial \theta}$, $R_t \equiv \frac{\partial R}{\partial t}$, $p_\tau \equiv \frac{\partial p}{\partial \tau}$, $p_\psi \equiv \frac{\partial p}{\partial \psi}$, $p_\theta \equiv \frac{\partial p}{\partial \theta}$, $p_t \equiv \frac{\partial p}{\partial t}$.

For these figures, after quite a long calculation, we have proved

Proposition 1. *The Jacobi Matrix of all GTR^n_m bodies (according to representation (2) for all fixed values of time t_0 has determinant*

$$det(J(\tau, \psi, \theta, t_0)) = p_\tau \cdot [R + p \cdot \cos(\psi + \mu \cdot g(\theta))][p - R_\psi \cdot \sin(\psi + \mu \cdot g(\theta))]. \tag{3}$$

Corollary 1. *If $R_\psi = 0$ and $R > p$, i.e. 1. function $R(\psi, \theta)$ is independent of argument ψ; and 2. the large radius R is greater than the p radius of radial cross section of GRT_m^n bodies (both are completely natural conditions), then representation (2) is a one to one correspondence of the points of the parallelogram (1) and points of corresponding GRT_m^n body.*

Proof. According to expression (3)

$$det(J(\tau, \psi, \theta, t_0)) = p \cdot p_\tau \cdot [R + p \cdot \cos(\psi + \mu \cdot g(\theta))] \neq 0.$$

1. $p_\tau \neq 0$ - natural condition (the opposite would mean that the function is independent of argument τ and this is impossible).
2. $p \neq 0$ - otherwise this means the radial cross section is a point!
3. $R + p \cdot \cos(\psi + \mu \cdot g(\theta)) > 0$ because $R > p > 0$. □

Remark 1. A) At any fixed moment t_0, this is a representation of a real three-dimensional body, which can be theoretically constructed from a parallelogram (1) by continuous deformation. This GRT_m^n or GML_m^n object (2) has no self-intersecting points.

B) The representation (2) can be considered as a complex motion of a three-dimensional body or plane surface in time as a superposition of elementary displacements. (Example in representation (2), replace argument θ with argument t and the functions do not depend on argument θ).

C) Roughly speaking the first movement is a twisting in a plane perpendicular to the main movement (the simplest example is the torsion of an aircraft propeller)!

D) The GML_m^n surfaces and bodies are a very important subset of GRT_m^n bodies ($Q(\theta, t) = 0$ or $Q(\theta, t)$ - is a 2π- periodic sufficiently smooth function of argument θ).

Proposition 2. ***1.*** *If $gcd(m, n) = k$, then a full external side of corresponding GML_m^n surface or body (Def.1.5 [2]) (with radial cross section convex polygon) is a k - colored surface; i.e. it is possible to paint the surface of this figure in k different colors without taking away of the brush. It is prohibited to cross the rib of this figure.*

2. *If $gcd(m, n) = k$, then a full side of corresponding GML_m^n surface or body (with radial cross section simple star [1]) is a $2k$ - colored surface;*

• **Some geometric properties of a “Regular” GML_2^n surfaces.**

This is one of the simplest but most important subclass of GML surfaces, when the shape of the basic line does not depend on arguments ψ, θ and $g(\theta) \equiv \theta$. They can be considered in two ways:

1. This is the static surface that is obtained from the rectangle by twisting n times around the axis of symmetry before gluing the ends (representation (4), when arguments ψ_0, t_0 are fixed). This means that the representation (2) has following simple form:

$$\begin{gathered} X(\tau, \psi_0, \theta, t_0) = T_1(t_0) + [R + \tau \cdot \cos(\psi_0 + \frac{n}{2} \cdot \theta)] \cos(\theta + M(t_0)), \\ Y(\tau, \psi_0, \theta, t_0) = T_2(t_0) + [R + \tau \cdot \cos(\psi_0 + \frac{n}{2} \cdot \theta)] \sin(\theta + M(t_0)), \\ Z(\tau, \psi_0, \theta, t_0) = T_3(t_0) + \tau \cdot \sin(\psi_0 + \frac{n}{2} \cdot \theta). \end{gathered} \tag{4}$$

2. This is a trace (trajectory surface) that a segment leaves when it revolves around a basic line perpendicular to it (particularly, representation (4*), when arguments ψ_0, θ_0 are fixed).

$$\begin{aligned} X(\tau,\psi_0,\theta_0,t) &= [R+\tau\cdot\cos(\psi_0+\frac{n}{2}\cdot t]\cos(\theta_0+t), \\ Y(\tau,\psi_0,\theta_0,t) &= [R+\tau\cdot\cos(\psi_0+\frac{n}{2}\cdot t]\sin(\theta_0+t)), \\ Z(\tau,\psi_0,\theta_0,t) &= \tau\cdot\sin(\psi_0+\frac{n}{2}\cdot t). \end{aligned} \tag{4*}$$

Remark 2. 1. According to Corollary 1, expression (3) and $p=\tau, p_\tau=1$,

$$det(J_{GML_2^n}(\tau,\psi_0,\theta,t_0)) = \tau\cdot[R+\tau\cdot\cos(\psi_0+\frac{n}{2}\cdot\theta)]$$

and this expression never vanishes, since the condition is always $R>\tau>0$ (because without this restriction it is impossible to make this object).

2. According to expression (4*) this trajectory may have some self-crossing points, because such a trajectory does not need such a restriction $R>\tau$.

3. According to Proposition 2 for GML_2^n surfaces:

a. If n is an even number, then each function X, Y, Z in the representations (4) or (4*) are 2π-periodic functions of the argument θ or t. ;

b. If n is a odd number, then each function X, Y, Z in the representation (4) is a 4π-periodic function satisfying the following properties (5) (Möbius property, see [3]).

$$X(\tau,\theta+2\pi);Y(\tau,\theta+2\pi);Z(\tau,\theta+2\pi) = X(-\tau,\theta);Y(-\tau,\theta);Z(-\tau,\theta). \tag{5}$$

Some examples and geometric properties of a "Semi-Regular" DRT_m^n and DML_m^n surfaces.

These are the trajectories of bodies or surface which appear when:

1. DRT_m^n - a plane m-symmetrical figure (or PR_m-prism) makes n-turns around the baseline (only in the tangent plane of the virtual cylinder) after one complete round-trip of this curve around the axis OZ;

2. DML_m^n - a plane m-symmetrical figure makes n-turns around the baseline (only in the tangent plane of the virtual cylinder) before gluing.

We will call these geometric objects Degenerated Rotated and Twisted DRT_m^n and Degenerated Möbius-Listing's DML_m^n surfaces or bodies. The analytic representations of these motions have the following form

$$\begin{aligned} X &= T_1 + R\cdot\cos(\theta+M(t)) + p\cdot\cos(\psi+\mu\cdot g(\theta))\cdot\sin(\theta+M(t)), \\ Y &= T_2 + R\cdot\sin(\theta+M(t)) + p\cdot\cos(\psi+\mu\cdot g(\theta))\cdot\cos(\theta+M(t)), \\ Z &= T_3 + Q(\theta,t) + p\cdot\sin(\psi+\mu\cdot g(\theta)). \end{aligned} \tag{6}$$

The main difference between this representation (6) and (2) is not only in mathematical form, but also in the value of the determinant of the Jacobi matrix (in this case it is sometime reset to zero).

Proposition 3. *The Jacobi matrix of all DTR_m^n bodies (according to representation (6)) for all fixed value of time t_0 has the determinant*

$$\begin{aligned} det(J_{DRT}(\tau,\psi,\theta,t_0)) = p \cdot p_\tau \cdot [R_\theta - \mu g_\theta R_\psi] \cdot [\sin^2(\theta + M(t)) - \cos^2(\theta + M(t))] \\ -p^2 \cdot p_\tau \cdot \cos(\psi + \mu \cdot g(\theta)) - R_\psi \cdot R \cdot p_\tau \cdot \sin(\psi + \mu \cdot g(\theta)) \\ +2p \cdot p_\tau \cdot \sin(\theta + M(t)) \cdot \cos(\theta + M(t)) \\ \times[R + R_\psi \cdot \sin(\psi + \mu \cdot g(\theta)) \cdot \cos(\psi + \mu \cdot g(\theta))]. \end{aligned} \tag{7}$$

Corollary 2. *Even the simplest case determinant of Jacobi matrix for some values of the argument is zero! This is the point of degeneration on the surface.*

$$det(J_{DRT}(\tau,\psi,\theta,t_0)) = p \cdot p_\tau \cdot [2R \cdot \sin(\theta + M(t)) \cdot \cos(\theta + M(t)) - p \cdot \cos(\psi + \mu \cdot g(\theta))].$$

Despite the conditions of the functions $R(\psi,\theta,t)$ and $p(\tau,\psi,\theta,t)$ there always remains the possibility that the determinants will be zero.

Studying the geometric properties of these surfaces is certainly possible, but this can only be done if necessary, and then only in those sub-domains where there are no points of degenerations.

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Received 26.05.2020; revised 28.07.2020; accepted 29.09.2020.

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