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ON THE SPACES OF SPHERICAL POLYNOMIALS AND GENERALIZED THETA-SERIES WITH QUADRATIC FORMS OF TYPE $(-2, q, 1)$

Ketevan Shavgulidze

Abstract. The spherical polynomials of order $\nu = 2$ with respect to quadratic form of type $(-2, q, 1)$ are constructed and the basis of the space of these spherical polynomials is established. The space of generalized theta-series with respect to the quadratic form of type $(-2, q, 1)$ is considered.

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1 Introduction.

Let

$$Q(X) = Q(x_1, \cdots, x_r) = \sum_{1 \le i \le j \le r} b_{ij} x_i x_j$$

be an integral positive definite quadratic form in an even number r of variables. That is, $b_{ij} \in \mathbb{Z}$ and $Q(X) > 0$ if $X \neq 0$. To $Q(X)$ we associate the even integral symmetric $r \times r$ matrix A defined by $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, where $i < j$. If $X = [x_1, \ldots, x_r]'$ denotes a column vector where $'$ denotes the transposition, then we have $Q(X) = \frac{1}{2} X' A X$. Let A_{ij} denote the algebraic adjunct of the element a_{ij} in $D = \det A$ and a^*_{ij} the corresponding element of A^{-1}. $\Delta = (-1)^{\frac{r}{2}} D$ denotes the discriminant of the quadratic form $Q(X)$; $\delta = gcd\Big(\frac{1}{2}A_{ii}, A_{ij}\Big)$ $(i, j = 1, 2, ..., r)$, $N = \frac{D}{\delta}$ is the step of the quadratic form $Q(X)$; $\chi(d)$ is a character of the quadratic form $Q(X)$, i.e. if Δ is square, then $\chi(d) = 1$. A positive quadratic form of weight $\frac{r}{2}$, step N and character χ is called a quadratic form of type $\Big(\frac{r}{2}, N, \chi\Big)$.

A homogeneous polynomial $P(X) = P(x_1, \cdots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1 \le i,j \le r} a^*_{ij} \Big(\frac{\partial^2 P}{\partial x_i \partial x_j} \Big) = 0 \tag{1}$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [1]), and

$$\vartheta(\tau, P, Q) = \sum_{n \in \mathbb{Z}^r} P(n) z^{Q(n)}, \qquad z = e^{2\pi i \tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im} \tau > 0$$

is the corresponding generalized r-fold theta-series.

Let $P(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$. Hecke [2] calculated the dimension of the space $P(\nu, Q)$, $\dim P(\nu, Q) = \binom{\nu+r-1}{r-1} - \binom{\nu+r-3}{r-1}$ and form the basis of the space of spherical polynomials of second order with respect to $Q(X)$.

Let $T(\nu, Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.,

$$T(\nu, Q) = \{\vartheta(\tau, P, Q) \,:\, P \in \mathcal{P}(\nu, Q)\}.$$

Gooding [1] calculated the dimension of the vector space $T(\nu, Q)$ for reduced binary quadratic forms Q. Gaigalas [3] gets the upper bounds for the dimension of the space $T(4, Q)$ and $T(6, Q)$ for some diagonal quadratic forms. In [4 - 6] we established the upper bounds for the dimension of the space $T(\nu, Q)$ for some quadratic forms of r variables, when $r = 3, 4, 5$, in a number of cases we calculated the dimension and form the bases of these spaces.

In this paper we form the basis of the space of spherical polynomials of second order $P(2, Q)$ with respect to some quadratic form $Q(X)$ of type (-2, q, 1) and obtained the space of generalized theta-series $T(2, Q)$ with spherical polynomial P of second order and quadratic form Q of type $(-2, q, 1)$.

2 The basis of the space $P(2, Q)$ and $T(2, Q)$.

Let

$$Q(X) = x_1^2 + x_2^2 + \frac{q+1}{3}x_3^2 + qx_4^2 + x_1x_2 + x_1x_3 + x_2x_3 + qx_3x_4,$$

where $q \equiv -1 (mod 6)$. $Q(X)$ be a quadratic form of type $(-2, q, 1)$ [7]. For these form

$$D = \det A = q^2, \qquad a_{11}^* = \frac{2(q-1)}{3q}, \qquad a_{12}^* = a_{21}^* = -\frac{(q+2)}{3q},$$

$$a_{13}^* = a_{31}^* = a_{23}^* = a_{32}^* = -\frac{2}{q}, \qquad a_{14}^* = a_{41}^* = a_{24}^* = a_{42}^* = \frac{1}{q},$$

$$a_{22}^* = -\frac{2(q+1)}{3q}, \qquad a_{33}^* = \frac{6}{q}, \qquad a_{34}^* = -\frac{3}{q}, \qquad a_{44}^* = \frac{2}{q}.$$

Let

$$P(X) = P(x_1, x_2, x_3, x_4) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} a_{kij} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} x_4^{j}$$

be a spherical function of order ν with respect to the positive quadratic form $Q(x_1, x_2, x_3, x_4)$ of four variables and

$$L = [a_{000}, a_{100}, a_{110}, a_{111}, a_{200}, \ldots, a_{\nu\nu\nu}]^T$$

be the column vector, where a_{kij} $(0 \leq j \leq i \leq k \leq \nu)$ are the coefficients of polynomial $P(X)$.

We have that $\dim P(\nu, Q) = \binom{\nu+3}{3} - \binom{\nu+1}{3}$. The polynomials (the coefficients of polynomial P_i are given in the brackets)

$$P_1(a_{000}^{(1)}, a_{100}^{(1)}, \dots, a_{\nu-2,\nu-2,\nu-2}^{(1)}, 1, 0, 0, \dots, 0),$$

$$P_2(a_{000}^{(2)}, a_{100}^{(2)}, \dots, a_{\nu-2,\nu-2,\nu-2}^{(2)}, 0, 1, 0, \dots, 0),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_t(a_{000}^{(t)}, a_{100}^{(t)}, \dots, a_{\nu-2,\nu-2,\nu-2}^{(t)}, 0, 0, 0, \dots, 1),$$

where the first $\binom{\nu+2}{4}$ coefficients from a_{000} to $a_{\nu-2,\nu-2,\nu-2}$ are calculated through other $\binom{\nu+3}{3} - \binom{\nu+1}{3} = t$ coefficients, form the basis of the space $P(\nu, Q)$.

The basis spherical polynomials of second order ($\nu = 2$) with respect to $Q(x_1, x_2, x_3, x_4)$ has the form:

$$P_1 = a_{000}^{(1)} x_1^2 + x_1 x_2, \qquad P_2 = a_{000}^{(2)} x_1^2 + x_1 x_3, \qquad P_3 = a_{000}^{(3)} x_1^2 + x_1 x_4,$$
$$P_4 = a_{000}^{(4)} x_1^2 + x_2^2, \qquad P_5 = a_{000}^{(5)} x_1^2 + x_2 x_3, \qquad P_6 = a_{000}^{(6)} x_1^2 + x_2 x_4,$$
$$P_7 = a_{000}^{(7)} x_1^2 + x_3^2, \qquad P_8 = a_{000}^{(8)} x_1^2 + x_3 x_4, \qquad P_9 = a_{000}^{(9)} x_1^2 + x_4^2.$$

Using (1) for P_i we have

$$P_1 = \frac{q+2}{2(q-1)} x_1^2 + x_1 x_2, \quad P_2 = \frac{3}{q-1} x_1^2 + x_1 x_3, \quad P_3 = -\frac{3}{2(q-1)} x_1^2 + x_1 x_4,$$

$$P_4 = \frac{q+1}{q-1} x_1^2 + x_2^2, \quad P_5 = \frac{3}{q+1} x_1^2 + x_2 x_3, \quad P_6 = -\frac{3}{2(q-1)} x_1^2 + x_2 x_4,$$

$$P_7 = -\frac{9}{q+1} x_1^2 + x_3^2, \quad P_8 = \frac{9}{2(q+1)} x_1^2 + x_3 x_4, \quad P_9 = -\frac{3}{q+1} x_1^2 + x_4^2.$$

They form the basis of the space of spherical polynomials of second order with respect to $Q(x)$ and $\dim P(2, Q) = \binom{5}{3} - \binom{3}{3} = 9$.

Now we construct the corresponding generalized theta-series

$$\vartheta(\tau, P_1, Q) = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} P_1(x) \Big) z^n = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} \Big(\frac{q+2}{2(q-1)} x_1^2 + x_1 x_2 \Big) \Big) z^n,$$

$$\vartheta(\tau, P_2, Q) = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} \Big(\frac{3}{q-1} x_1^2 + x_1 x_3 \Big) \Big) z^n,$$

$$\vartheta(\tau, P_3, Q) = \sum_{n=1}^{\infty} \Big(\sum_{Q(x)=n} (-\frac{3}{2(q-1)} x_1^2 + x_1 x_4) \Big) z^n,$$

$$\vartheta(\tau,P_4,Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\big(\frac{q+1}{q-1}x_1^2+x_2^2\big)\Big)z^n,\quad \vartheta(\tau,P_5,Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\Big(\frac{3}{q+1}x_1^2+x_2x_3\Big)\Big)z^n,$$

$$\vartheta(\tau,P_6,Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\Big(-\frac{3}{2(q-1)}x_1^2+x_2x_4\Big)\Big)z^n,\ \vartheta(\tau,P_7,Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\Big(-\frac{9}{q+1}x_1^2+x_3^2\Big)\Big)z^n,$$

$$\vartheta(\tau,P_8,Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\Big(\frac{9}{2(q+1)}x_1^2+x_3x_4\Big)\Big)z^n,\ \vartheta(\tau,P_9,Q)=\sum_{n=1}^{\infty}\Big(\sum_{Q(x)=n}\Big(-\frac{3}{q+1}x_1^2+x_4^2\Big)\Big)z^n.$$

This generalized theta-series form the system of generators of the space $T(2,Q)$. We have the following

Theorem. *The generalized theta-series:*

$$\vartheta(\tau,P_1,Q);\ \ \vartheta(\tau,P_2,Q);\ \ \cdots\ ;\ \ \vartheta(\tau,P_9,Q)$$

form the system of generators of the space $T(2,Q)$.

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Author(s) address(es):

Ketevan Shavgulidze
Department of Mathematics, Faculty of Exact and Natural Sciences
I. Javakhishvili Tbilisi State University
University str. 13, 0186 Tbilisi, Georgia
E-mail: ketevan.shavgulidze@tsu.ge