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ON THE NUMERICAL COMPUTATIONS OF AN ANTI-PLANE PROBLEM IN THE CASE OF ISOTROPIC COMPOSITE BODY WEAKENED BY A CRACK

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Abstract. The anti-plane problem of the elasticity theory for a composite (piecewise homogeneous) orthotropic (in particular, isotropic) plane weakened by a crack is reduced to a system of singular integral equations containing a fixed-singularity with respect to characteristic functions of disclosure of crack when the crack intersects the dividing border of interface with the right angle. The method of discrete singularity is applied to finding a solution of the obtained system. The corresponding new algorithm is constructed and realized. In this work, the behavior of the solutions is studied. The results of computations are represented.

Keywords and phrases: Singular integral equations, crack, anti plane problem, collocation method, method of discrete singularity, numerical realization.

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1 Introduction. Study of boundary value problems for the composite bodies weakened by cracks has a great practical significance. In the articles [1] and [2] an anti-plane problem of the elasticity theory for composite (piece-wise homogeneous) orthotropic body weakened by crack when the crack intersect an interface or reaches it at right angle are studied. The studied problems are reduced to the singular integral equation (when crack reaches to the interface) and system (pair) of singular integral equations (when crack intersects the interface) which in both cases contains an fixed-singularity, with respect to the unknown characteristic functions (the tangent stress jumps) of disclosure of cracks. In the present article the problems for composite (piece-wise homogeneous) bodies when cracks intersect an interface at right angle is studied. Behavior of solutions in the neighborhood of the crack endpoints is studied by a method of discrete singularity with uniform division of an interval by knots.

2 Statement of the problem. The system of singular integral equations with respect to leaps $\varrho_k(x)$ (see [1, 2])

$$\begin{aligned} &\int_0^1 \left(\frac{1}{t-x} - \frac{a_1}{t+x}\right)\varrho_1(t)dt + b_1\int_{-1}^0 \frac{\varrho_2(t)dt}{t-x} = 2\pi f_1(x), \quad x \in (0;1), \\ &b_2\int_0^1 \frac{\varrho_1(t)dt}{t-x} + \int_{-1}^0 \left(\frac{1}{t-x} - \frac{a_2}{t+x}\right)\varrho_2(t)dt = 2\pi f_2(x), \quad x \in (-1;0), \end{aligned} \tag{1}$$

where $\varrho_k(x)$, $f_k(x)$ unknown and given and given real functions, respectively, a_k, b_k constants,

$$a_k = \frac{1-\gamma_k}{1+\gamma_k}, \quad b_k = \frac{2}{1+\gamma_k}, \quad \gamma_1 = \frac{1}{\gamma_2}, \quad \gamma_2 = \frac{b_{55}^{(2)}}{b_{55}^{(1)}}, \quad f_k(x) = \frac{\lambda_k}{b_{44}^{(k)}} q_k(x),$$

$f_k(x) \in H$, $\varrho_k(x) \in H^*$, $k = 1, 2$; $\lambda_k^2 = \dfrac{b_{44}^{(k)}}{b_{55}^{(k)}}$, $b_{44}^{(k)}$, $b_{55}^{(k)}$ are elastic constants, $q_k(x)$ is a function of Holder's class, $k = 1, 2$; In particular, if we have isotropic case $b_{44}^{(k)} = b_{55}^{(k)} = \mu_k$, $\lambda_k = 1$, where μ_k is modulus of displacement, $k = 1, 2$.

Exploration of behavior of solutions near the ends of the boundary presents a special interest. The solutions of the system (1) of the integral equations can be presented in the following way:

$$\varrho_1(t) = \frac{\chi_1(t)}{t^{\alpha_1}(1-t)^{\beta_1}}, \qquad \varrho_2(t) = \frac{\chi_2(t)}{t^{\alpha_2}(1+t)^{\beta_2}},$$

where α_k, β_k are unknown constants $0 < \alpha_k$, $\beta_k < 1$, and $\chi_k(t)$ are functions, which belong to Holder's class, $k = 1, 2$. In the point $t = \pm 1$ we obtain correspondingly $\beta_1 = \beta_2 = \dfrac{1}{2}$. In the considered case there is no singularity in the point $t = 0$ (see [1]).

3 Algorithm. Now let's consider the system of the singular integral equations (1) containing fixed singularity. The system (1) of singular integral equations is solved by a collocation method, in particular, a method discrete singularity (see [3]). Let us first of all consider an algorithm of uniform division. Solutions system of equations (1) have such view $\varrho_1(t) = \dfrac{\varrho_1^*(t)}{\sqrt{1-t}}$, $\varrho_2(t) = \dfrac{\varrho_2^*(t)}{\sqrt{1+t}}$ (see [2]). Let's enter such distribution of knots for variables of integration and account points accordingly: $t_{1i} = 0 + i \cdot h$, $t_{2i} = -1 + i \cdot h$, $i = 1, 2, \cdots, n$; $x_{1j} = t_{1j} - \dfrac{h}{2}$, $x_{2j} = t_{2j} + \dfrac{h}{2}$, $j = 1, 2, \cdots, n$; $h = \dfrac{1}{n+1}$. The pair of the equations (1) probably to present as follows with the help quadrature formulas [2],

$$\begin{aligned}
&\sum_{i=1}^{n} \left(\frac{h}{t_{1i} - x_{1j}} - \frac{a_1 h}{t_{1i} + x_{1j}} \right) \varrho_1(t_{1i}) + b_1 \sum_{i=1}^{n} \left(\frac{h}{t_{2i} - x_{1j}} \right) \varrho_2(t_{2i}) = 2\pi f_1(x_{1j}), \\
&j = 1, 2, \cdots, n, \\
&b_2 \sum_{i=1}^{n} \left(\frac{h}{t_{1i} - x_{2j}} \right) \varrho_1(t_{1i}) + \sum_{i=1}^{n} \left(\frac{h}{t_{2i} - x_{2j}} - \frac{a_2 h}{t_{2i} + x_{2j}} \right) \varrho_2(t_{2i}) = 2\pi f_2(x_{2j}), \\
&j = 1, 2, \cdots, n.
\end{aligned} \tag{2}$$

Thus, we have $2n$ equations with $2n$ unknowns. The received system of the linear algebraic equations (2) is possible to solve with the help to one of direct method, for example, by the Gauss modified method.

4 The numerical realization. For the approximate solving of the system of singular integral equations (1) the several programs in "Maple" is composed. The algorithm has been approved by tests and the results of numerical computations are represented in tables. In the above-mentioned research problem our main objective was investigation of a possible distribution of cracks along a body, study behavior of solution (the character function of stress) and finding the coefficients of intensity of stress (cis_1, cis_2) in a vicinity at the ends of the cracks. As we have mentioned a main objective was studying of behavior of solution in a vicinity at the ends of cracks and a finding the coefficients of intensity of stress. For this purpose we have calculated values of the coefficients of intensity of stress $cis_1 = \lim\limits_{x\to -1}\sqrt{1+x}\varrho_2(x)$, $cis_1 \approx \sqrt{1+x_{21}}\varrho_2(x_{21})$, $cis_2 = \lim\limits_{x\to +1}\sqrt{1-x}\varrho_1(x)$, $cis_2 \approx \sqrt{1-x_{1n}}\varrho_1(x_{1n})$, at the ends of cracks by using algorithm of interval [-1,+1] uniform splitting and on each step of calculations increase number of division of an interval two times. The approached values of coefficients of intensity of stresses (stress intensity factor) in vicinity of the ends of cracks. Consider a body, composed of two isotropic materials (copper and aluminum). For copper $b_{44}^{(k)} = b_{55}^{(k)} = 45.5$ GPa, and for aluminum $b_{44}^{(k)} = b_{55}^{(k)} = 25.5$ GPa. Numerical calculations for the functions $q_1(x)$ and $q_2(x)$ we are studying the following three cases: $q_1(x) = 0.01$, $q_2(x) = 0.01$ (variant 1), $q_1(x) = 0.01$, $q_2(x) = 0.02$ (variant 2), $q_1(x) = 0.02$, $q_2(x) = 0.01$ (variant 3), are given by the table. The base system of singular integral equations (1) is solved by a discrete singular method in the interval [-1,1] when the number of the crack splitting are $n = 16$, $n = 32$, $n = 64$.

variant	$cis \setminus n$	16	32	64
1	cis_1	-0.00031560	-0.00034338	-0.00036936
	cis_2	0.00003067	0.00001863	0.00000589
2	cis_1	-0.00079752	-0.00088447	-0.00096753
	cis_2	-0.00013563	-0.00017907	-0.00022287
3	cis_1	-0.00014929	-0.00014567	-0.00014060
	cis_2	0.00022765	0.00023498	0.00024056

If the absolute values of the coefficients of intensity of tension are less than 1 (it to very close to critical limit of distribution of a crack) but it is close to unit then cracks spread slowly. If the absolute values of the coefficients of intensity of tension are considerably less than 1, then cracks almost do not develop (see [4]). Numerical experiments have shown that increment of loading at the ends of the crack causes increment of values of the coefficients of the intensity of tension. As we consider linear tasks of the elasticity theory so increment or diminution loading will lead to proportionally increment or diminution of values of relevant solutions. The last fact gives possibility to make the hypothetical forecasts about developments of a crack.

R E F E R E N C E S

1. Papukashvili, A. Antiplane problems of theory of elasticity for piecewise-homogeneous orthotropic plane slackened with cracks. *Bulletin of the Georgian Academy of Sciences, Tbilisi*, **169**, 2 (2004), 267-270.

2. Papukashvili, A., Davitashvili, T., Vashakidze, Z. Approximate solution of anti-plane problem of elasticity theory for composite bodies weakened by cracks by integral equation method. *Bulletin of the Georgian National Academy of Sciences*, **9**, 3 (2015), 50-57.

3. Belotserkovski, S.M., Lifanov, I.K. Numerical methods in the singular integral equations and their application in aerodynamics, the elasticity theory, electrodynamics. *Nauka, Moscow*, (1985), 256.

4. Aleksandrov, V.M., Mkhitarian, S.M. Contact problems for bodies with thin coverings and lauers. *Nauka, Moscow*, (1983), 488.

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