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FOURIER TRIGONOMETRIC SERIES WITH GAPS

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Abstract. The sufficient conditions of the generalized absolute convergence of Fourier trigonometric series with gaps are established for some classes of functions.

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1 Introduction. It is known that the principle of localization for absolute convergence of Fourier series of the function $f \in L(-\pi;\pi)$, generally speaking, is invalid [1, p. 638]. But if, in addition, f has a Fourier series of the type

$$\sum_{k=1}^{\infty}(a_{n_k}\cos n_k x + b_{n_k}\sin n_k x),$$

where

$$\lim_{k\to\infty}\frac{\min(n_k-n_{k-1};n_{k+1}-n_k)}{\ln n_k}=\infty,$$

then as Noble [6] derived a number of properties of the coefficients a_{n_k}, b_{n_k} under various hypotheses about the behaviour of f in an arbitrary small interval take place.

Then Kennedy proved [5] that Noble's theorems remain valid under the unique assumptions $n_{k+1}-n_k\to\infty$ as $k\to\infty$.

2 Content. Below assume, that $f \in L(-\pi;\pi)$ is a real function periodic with period 2π and Fourier series of f is given by

$$f(x)\sim\sum_{k=1}^{\infty}(a_{n_k}\cos n_k x + b_{n_k}\sin n_k x),$$

where a_{n_k}, b_{n_k} are Fourier coefficients of the function f and $\lim\limits_{k\to\infty}(n_{k+1}-n_k)=\infty$.

The problem of convergence of the series

$$\sum_{k=1}^{\infty}\gamma_k\rho_{n_k}^r(f),\quad 0<r<2,$$

is considered, where $\rho_{n_k}(f)=\sqrt{a_{n_k}^2+b_{n_k}^2}$ and $\gamma=\{\gamma_k\}$, $k\in\mathbb{N}$, is the sequence of nonnegative numbers, satisfying definite conditions.

A sequence $\gamma = \{\gamma_k \geq 0\}$, $k \in \mathbb{N}$, is said [2] to belong to the class A_α for some $\alpha \geq 1$, if

$$\Big(\sum_{k\in D_\mu} \gamma_k^\alpha\Big)^{\frac{1}{\alpha}} \leq C \cdot 2^{\mu\,\frac{1-\alpha}{\alpha}} \sum_{k\in D_{\mu-1}} \gamma_k, \quad \mu \in \mathbb{N},$$

where $D_0 = \{1\}$, $D_\mu = \{2^{\mu-1}+1, \dots, 2^\mu\}$, $\mu \in \mathbb{N}$, and the constant C does not depend on μ.

$M(I)$ denotes a class of bounded functions on the segment $I = [a;b]$.

$\mathrm{BV}_s(I)$ is the class of the functions with bounded s-variation on the I.

The modulus of δ-variation of the function $f \in M(I)$ is denoted by $\varphi(n;\delta;f;I)$ and its definition was introduced by Karchava [4], according to Chanturia's modulus of variation in the following way:

$$\varphi(0;\delta;f;I) = 0 \ \text{ for any } \delta > 0,$$

and

$$\varphi(n;\delta;f;I) = \sup_{\Pi_{n,\delta}} \sum_{k=1}^{n} \omega(f;I_k), \quad k \in N,$$

where $\Pi_{n,\delta}$ is a system consisting of n nonintersecting intervals $\{I_k\}$ of the segment I. The length of each interval I_k is equal to δ, and $\omega(f;I_k)$ is the oscillation of the function f on I_k.

The following statement is true.

Theorem. *Let $f \in M(I)$, $I = [x_0-\delta_1; x_0+\delta_1]$ be a proper subset of the interval $T = (-\pi;\pi)$, $\{\gamma_k\} \in A_{\frac{2}{2-r}}$ for some $0 < r < 2$, then*

$$\sum_{k=1}^{\infty} \gamma_k \rho_{n_k}^r(f) \leq C \sum_{k=1}^{\infty} \gamma_k k^{-r} \Big(\sum_{j=1}^{k} \frac{\varphi^2(j;\frac{1}{k};f;I)}{j^2}\Big)^{\frac{r}{2}}.$$

The Theorem for $r=1$, $\gamma_k = 1$ leads to:

Corollary 1. *If $f \in M(I)$, then*

$$\sum_{k=1}^{\infty} |a_{n_k}| + |b_{n_k}| \leq C \sum_{k=1}^{\infty} \frac{1}{k} \Big(\sum_{j=1}^{k} \frac{\varphi^2(j;\frac{1}{k};f;I)}{j^2}\Big)^{\frac{1}{2}}.$$

Corollary 1 was obtained by us earlier [7] and it was shown that from this corollary follows Noble's and Kennedy's theorems about the absolute convergence of Fourier series with gaps.

Corollary 2. *Let $f \in C(I) \cap \mathrm{BV}_s(I)$, $s \in [1;2]$, $\{\gamma_k\} \in A_{\frac{2}{2-r}}$, $0 < r < 2$, then*

$$\sum_{k=1}^{\infty} \gamma_k \rho_{n_k}^r(f) \leq C \sum_{k=1}^{\infty} \gamma_k k^{-r} \omega^{(2-s)\,\frac{r}{2}}\Big(f;\frac{1}{k};I\Big),$$

where $\omega(f;\frac{1}{k};I)$ denotes the modulus of continuity of a function f on the segment I.

Corollary 2 presents the analogue of the theorem, obtained by Gogoladze L. and Meskhia R. [3] for Fourier series with gaps.

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