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RENORMDYNAMICS OF SPACE DIMENSION AND QUARKONIUM POTENTIALS

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Abstract. Scale dependent space dimension models for quarkonium are considered. Confining potential including topological fluctuations of the vacuum constructed.

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Quarkonium spectroscopy indicates that between valence quarks inside hadrons, the potential on small scales has $D = 3$ Coulomb form and at hadronic scales has $D = 1$ Coulomb one. We may combine this two types of behavior and form an effective potential in which at small scales dominates the $D = 3$ component and at hadronic scale the $D = 1$ component: the Coulomb-plus-linear potential (the "Cornell potential" [1]),

$$V(r) = -\frac{k}{r} + \frac{r}{a^2} = \mu\left(x - \frac{k}{x}\right), \ \mu = 1/a = 0.427\, GeV, \ x = \mu r, \tag{1}$$

where $k = \frac{4}{3}\alpha_s = 0.52 = x_0^2, \ x_0 = 0.72$ and $a = 2.34\, GeV^{-1}$ were chosen to fit the quarkonium spectra.

An important step in the solution of a theoretical problem is to find a good initial approximation in the corresponding mathematical model. Then by small deformations and a few terms in perturbation expansion we describe a physical phenomenon. When a deformation parameter (e.g. coupling constant) value increases, in some region the initial approximation might change into a new form. In the case of QCD, the coupling constant increases with increasing distance between quarks, and in the intermediate region (~ 0.5 fm) the three dimensional hadronic space becomes a fractal - a space with intermediate dimension. At the hadronic scale (~ 1 fm) we again have a nice classical picture (one dimensional space) and already one gluon exchange between valence quarks gives a confining potential. In the paper [2] we extend investigations started in [3] and construct such potentials and effective dimensions as functions of r.

Let us take one of the dimensions y as a circle with radius R. This corresponds to the periodic structure with a point charge sources at each point $y_n = y + 2\pi Rn, n = 0, \pm1, \pm2, ...$

$$\Delta\varphi = e\sum_n \delta^D(x)\delta(y_n), \varphi(D, r, y) = \sum_n \varphi(D, r, y_n),$$
$$V(D, r, y) = -\alpha(D+1)\sum_{n=-\infty}^{\infty}(r^2 + (2\pi Rn + y)^2)^{(1-D)/2}. \tag{2}$$

When $D = 3$, the potential (2) can be writen in a closed form [4]

$$V_3(r, y) = -\frac{\alpha(4)}{2Rr}\frac{\sinh(r/R)}{\cosh(r/R) - \cos(y/R)} = \begin{cases} \alpha(4)/(2Rr), & r \gg R, \\ \alpha(4)/(r^2 + y^2), & r,\ y \ll R, \end{cases} \tag{3}$$

where $\alpha(4)/(2R) = \alpha(3)$. We have the following expansion for V_3, [5]

$$V_3(r, \theta) = \sum_{n \geq 0} V_n(r, 3)\cos(n\theta),\ V_n(r, 3) = -\frac{\alpha(3)e^{-nr}}{r},\ \theta = y/R. \tag{4}$$

Now we show an extension of this relation for D dimensions. For this we consider **point charge Poisson equation of a massive particle and Yukawa potential**

$$\Delta V - m^2 V = e^2\delta^D(x). \tag{5}$$

Let us test the following Yukawa potential as a solution

$$\begin{aligned} &V(r) = -\frac{\alpha_D}{r^{D-2}}e^{-mr} = V_0 e^{-mr},\ \Delta V_0 = e^2\delta^D(x),\ \Delta = \frac{d^2}{d^2r} + \frac{D-1}{r}\frac{d}{dr}, \\ &\Delta V - m^2 V = e^{-mr}\frac{d^2}{dr^2}V_0 + 2(-\frac{D-2}{r})(-m)V + m^2 V \\ &+e^{-mr}\frac{D-1}{r}\frac{d}{dr}V_0 + \frac{D-1}{r}(-m)V - m^2 V \\ &= e^{-mr}\Delta V_0 + m\frac{(2(D-2) - (D-1))}{r}V = e^2\delta^D(x) + \frac{m(D-3)}{r}V. \end{aligned} \tag{6}$$

The second term in the right hand side is zero in $D = 3$. For $r >> 1/m, D \neq 3$, if we neglect the second term, the Yukawa potential will be approximate solution. We can extract from this calculation also the following result: **the D - dimensional Yukawa potential V_D is exact solution for the following point charge problem:**

$$\begin{aligned} &\Delta V - m^2 V - \frac{m(D-3)}{r}V = e^2\delta^D(x), \\ &V_D(r) = -\frac{\alpha_D}{r^{D-2}}e^{-mr} = V_0 e^{-mr},\ \Delta V_0 = e^2\delta^D(x),\ \Delta = \frac{d^2}{d^2r} + \frac{D-1}{r}\frac{d}{dr}. \end{aligned} \tag{7}$$

Having exact equation for Yukawa potentials, we may formulate and solve the problem of point charge in $D + 1$ dimensional space with one compact dimension in the following way

$$\begin{aligned} &\Big(\Delta V - n^2 m^2 V - \frac{nm(D-3)}{r}V\Big)e^{inmy} \\ &= \Big(\Delta_D + \frac{(D-3)}{r}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\Big)V_{m_n}(r)e^{inmy} = e^2\delta^D(x)e^{inmy}, \\ &m = 1/R,\ n = 0, \pm 1, \pm 2, ..., \\ &\Big(\Delta_{D+1} + \frac{(D-3)}{r}\frac{\partial}{\partial x_{D+1}}\Big)V(x) = e^2\delta^{D+1}(x),\ x_{D+1} = y = R\theta,\ my = \theta, \end{aligned}$$

$$
\begin{aligned}
&\Delta_{D+1} + \frac{(D-3)}{r}\frac{\partial}{\partial x_{D+1}} = \Delta_D + \Big(\frac{\partial}{\partial x_{D+1}} + \frac{(D-3)}{2r}\Big)^2 + \frac{(D-3)(5-D)}{4r^2},\\
&V(x) = \sum_n V_n(r)e^{in\theta} = V_0(r)\Big(1 + \sum_{n\geq 1} e^{-nmr}\cos(n\theta)\Big)\\
&= V_0\Big(1 + \frac{e^{-mr+i\theta}}{1-e^{-mr+i\theta}} + \frac{e^{-mr-i\theta}}{1-e^{-mr-i\theta}}\Big)\\
&= \frac{V_0(1-e^{-2mr})}{1+e^{-2mr}-2e^{-mr}\cos\theta} = \frac{V_0\sinh(mr)}{\cosh(mr)-\cos\theta},\\
&V_n = V_0e^{-nmr},\ V_1 = V_0e^{-mr},\ m = 1/R,\ V_0 = -\alpha_D r^{2-D}.
\end{aligned}
\tag{8}
$$

Note that in the repulsive case we have also the following confining potential

$$
V(r) = \frac{\alpha_D}{r^{D-2}}e^{mr} = V_0e^{mr} = \frac{\alpha_D m^{D-2}}{x^{D-2}}e^x = \alpha x^{2-D}e^x,\ x = mr. \tag{9}
$$

Now we consider **nonlinear point charge problem in extended quantum mechanics**. In the extended quantum mechanics [6] the point charge problem is

$$
iV_t - \Delta V + \frac{1}{2}V^2 = -e^2\delta^D(x), \tag{10}
$$

which is reduced to the nonlinear Poisson equation in the static case

$$
\Delta V - gV^2 = e^2\delta^D(x), \tag{11}
$$

where we introduce new coupling constant g. When $g = 0$, we have ordinary point charge problem with the solution $V = V_c \sim r^{2-D}$. In the sourceless case, $e = 0$, we have the solution $V_n = 2(4-D)/gr^{-2}$. If we take $V = V_c + V_n + U$, for U, we find

$$
\Delta U - 2g(V_c + V_n)U = g(V_c + V_n)^2. \tag{12}
$$

In the case of three dimension $D = 3$, on the small scales dominates the nonlinear repulsive solution $V = 2/gr^{-2},\ g > 0$. For large scales dominates the Coulomb attractive solution $V = -\alpha/r$.

In [7] by proper account of the compact nature of SU(3) gauge group that gives rise to the periodic θ-vacuum of the theory, the gluon propagator was modified as

$$
G(p) = (p^2 + \chi/p^2)^{-1} = \frac{p^2}{p^4+\chi} = \frac{1}{2}\Big(\frac{1}{p^2 + i\sqrt{\chi}} + \frac{1}{p^2 - i\sqrt{\chi}}\Big), \tag{13}
$$

which gives the potential

$$
\begin{aligned}
&V(r) = -\frac{\alpha\cosh\mu r\cos\mu r}{r} = -\frac{\mu\alpha\cosh x\cos x}{x} = \mu\alpha\Big(-\frac{1}{x} + \frac{x^3}{6} + \ldots\Big),\\
&x = \mu r,\ \mu = \sqrt[4]{\chi}/\sqrt{2},
\end{aligned}
\tag{14}
$$

where χ is the Yang-Mills topological susceptibility related to the η' mass by the Witten - Veneziano relation,

$$\chi = \frac{F_\pi^2}{2N_f}(m_{\eta'}^2 + m_\eta^2 - 2m_K^2) \simeq (180MeV)^4, \ \mu = \sqrt[4]{\chi}/\sqrt{2} = 127MeV. \tag{15}$$

The topological susceptibility in this formula is the only quantity which is by definition calculable in gluodynamics. Early papers of its calculation are [8-10] more recent [11].

The potential (14) is well motivated and confining. In the minimum of the potential bound states "bags" have size of the order of 11 fm,

$$r = 7/\mu = 7/0.127GeV^{-1} = 11fm, \, GeV^{-1} \simeq 0.2fm\,, \tag{16}$$

and can give rise to long lived states corresponding to hadronic halos or galactic (in case of gravitational) halos.

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