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SYMMETRIC FOLIATIONS IN SPACES OF TRIANGLES

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Abstract. We discuss the geometric structure of certain natural foliations in the moduli spaces of Euclidean triangles.

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1 Introduction. The aim of this paper is to present several results on certain foliations in the moduli spaces of Euclidean triangles introduced in [1]. Following [1] we consider the set $X_\triangle$ which is defined as the closure of the open subspace $D_\triangle$ of the first octant $\mathbb{R}^3_+$ defined by the triangle inequalities: $x < y+z, y < x+z, z < x+y$. The spaces $X_\triangle$ and $D_\triangle$ are called the *closed space of triangles* and *open space of triangles* respectively. Thus by definition each point $(a,b,c) \in X_\triangle$ corresponds to a triangle $T(a,b,c)$ which is defined up to congruence. Notice that some of vertices of $T(a,b,c)$ may coincide, in which case we speak of *a degenerate triangle*.

We also consider several differentiable functions on $D_\triangle$, defined by symmetric functions of the sides of triangle and introduce foliations of $D_\triangle$ by their level surfaces. Let P be the perimeter of $T(x,y,z)$, S the area of triangle $T(x,y,z)$, R the circumradius of $T(x,y,z)$, by r the inradius of $T(x,y,z)$, and by E the *electrostatic energy* of unit charges placed at vertices of $T(x,y,z)$ defined as $E(x,y,z) = x^{-1} + y^{-1} + z^{-1}$.

Notice that all these functions are differentiable, symmetric in x,y,z and positive on $D_\triangle$. For the function $f : D_\triangle \to \mathbb{R}$ and $c \in \mathbb{R}$, by f_c we denote the level surface $\{f = c\}$ in $D_\triangle$ and call it f-level. We will deal with foliations F_P, F_S, F_R, F_r, F_E of $D_\triangle$ by level surfaces of functions P, S, R, r, E. It is easy to verify that all levels of these functions in $D_\triangle$ are smooth two-dimensional (2d) surfaces. For any two functions f, g we will also consider the intersections $f_c \cap g_d$. It can be shown that, for functions P, S, R, r, E, any non-empty intersection of such type is either a smooth closed curve or a single point. For area and perimeter these facts have been proven in [1]. The polynomial $H(x,y,z) = (x+y+z)(-x+y+z)(x-y+z)(x+y-z)$ will be called *Heron polynomial*. If $S = S(x,y,z)$ then by Heron's formula one has $16S^2 = H(x,y,z)$. It is easy to verify that $H(x,y,z) = -\sigma_1^4 + 4\sigma_1^2\sigma_2 - 8\sigma_1\sigma_3$, where $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric functions of x,y,z.

In the sequel we establish further geometric properties of these foliations and associated extremal problems analogous to the classical isoperimetric problem [2]. More precisely, we fix the values of two of the aforementioned functions and search for the constrained extrema of the third one from the same list. It should be noted that the proofs make essential use of representation of functions S, E, R, r, H in terms of elementary symmetric functions of the sides, which is one of the peculiar features of our approach.

2 Geometric properties of area and Coulomb foliations. The following property of area foliation presented in [3] is based on certain observations given in [1] without proof.

Theorem 1. *(see [3]) Area levels in $D_\triangle$ are smooth convex two-dimensional surfaces in the open space of triangles $D_\triangle$.*

A rigorous proof of this result based on the investigation of Gaussian curvature can be found in [3]. Later on, the present author found a simpler proof, based on properties of real plane cubic curves. We outline the new argument here since it also yields an analogous result presented in the next section. Consider a non-empty intersection $S_c \cap P_d = \{S = c\} \cap \{P = d\}$. From Heron's formula follows that $S_c \cap P_d$ is a component of a plane cubic curve. The smoothness of this curve was shown in [1]. So $S_c \cap P_d$ is a smooth compact component of a plane cubic curve. It follows that any straight line in the ambient plane intersects this set in no more than two points, which implies convexity of the closed curve $S_c \cap P_d$. Using the latter fact it is easy to derive convexity of the whole area level $\{S = c\}$ in $D_\triangle$ by a standard geometric argument.

Proposition 1. *The set $S_c \cap P_d$ is non-empty if and only if $36c \leq d^2\sqrt{3}$. If $36c = d^2\sqrt{3}$ this intersection consists of one point representing the class of regular triangle.*

This follows from the well known fact that the regular triangle of perimeter P has the maximal area among all triangles of the same perimeter P. Since the class of a regular triangle is the unique critical point of perimeter on the convex surface, the following result mentioned in [3] follows by standard arguments of Morse theory.

Theorem 2. *(see [3]) The negative gradient flow of perimeter on an area level $S = c$ carries each point of $S = c$ to the class of a regular triangle.*

As a curious geometric corollary we obtain that any non-regular triangle can be continuously deformed to a regular triangle so that the area remains constant and perimeter is monotonously decreasing.

Proposition 2. *The domain, bounded by a non-empty set $S_c \cap P_d$, contains the representative of a regular triangle with the side equal to $d/3$.*

This is obvious since the bisector of the first octant intersects the P-level $\{P = d\}$ at the point $(d/3, d/3, d/3)$.

Proposition 3. *Each non-degenerate curve $S_c \cap P_d$ contains (representatives of) two non-congruent isosceles triangles.*

Notice that each non-empty curve $S_c \cap P_d$ encircles the representative of a regular triangle which coincides with the center of a regular triangle equal to P-level in the closed space of triangles $X_\triangle$. By Theorem 1 any non-empty curve $S_c \cap P_d$ is convex and so the bisector of each angle of the mentioned regular triangle intersects $S_c \cap P_d$ in two points which correspond to two (non-congruent) isosceles triangles.

Our constructions yield analogous results for other aforementioned fibrations. As an example we consider fibration F_E which for brevity is called *Coulomb fibration*.

Proposition 4. *Any non-empty and non-degenerate intersection of the form $E_c \cap P_d$ is a smooth closed curve in the plane $\{x + y + z = d\}$.*

It is easy to verify that gradients of E and P are linearly independent at any point of $E_c \cap P_d$, which implies the result.

Theorem 3. *Levels of Coulomb fibration in $D_\triangle$ are smooth convex two-dimensional surfaces in the open space of triangles $D_\triangle$.*

Smoothness of Coulomb levels E_c is obvious since the gradient $\nabla E = (-x^{-2}, -y^{-2}, -z^{-2})$ does not vanish at any point of $D_\triangle$. Obviously, the intersection $E_c \cap P_d$ is defined by the equation $x^{-1} + y^{-1} + (d - x - y)^{-1} = c$ which is equivalent to a cubic equation in (x, y). By Proposition 4 the curve $E_c \cap P_d$ is the smooth compact component of a plane cubic curve and so one can complete the proof using the same argument as in Theorem 1. The next result is derived from Theorem 3 using the same arguments from Morse theory as in the proof of Theorem 2.

Theorem 4. *The negative gradient flow of perimeter on a Coulomb level $E = c$ carries each point of $E = c$ to the class of regular triangle.*

3 Symmetric coordinates in spaces of triangles. Our results can be used to show that values of some triples of the symmetric functions considered above uniquely determine the shape of the triangle and thus define new coordinate system in the space of triangles. For triple (P, S, E) this was established in [3]. We extend the result given in [3] as follows.

Theorem 5. *Each of triples $(P, S, E), (P, R, S), (P, R, E), (P, R, r)$ defines a coordinate system in the space $D_\triangle$.*

The proof uses the same method as in [3]. In each of four cases the formula for Heron polynomial given in the Introduction combined with some high school geometry enables one to obtain a system of three equations for the elementary symmetric functions $\sigma_1, \sigma_2, \sigma_3$ of the sidelengths of the sought triangle. It turns out that each of the arising systems has only one real solution defined up to the order of variables, which means that it uniquely determines the class of the sought triangle in $D_\triangle$ and yields the result. For example, for the triple (P, R, S) this system is:

$$\{\sigma_1 = P, \sigma_3 = 4RS, -\sigma_1^4 + 4\sigma_1^2\sigma_2 - 8\sigma_1\sigma_3 = 16S^2\}.$$

Inserting the first two expressions in the third equation one gets a linear equation for σ_2 and it becomes obvious that there is a unique triple $(\sigma_1, \sigma_2, \sigma_3)$ satisfying this system. Application of the same argument in the remaining cases completes the proof of Theorem 5.

Our results on the area and Coulomb foliation also enable one to get information on the image of $D_\triangle$ under the map defined by the new coordinates. Necessary conditions on the value of symmetric coordinates can be obtained by solving the extremal problem for each of the functions for fixed values of two other functions. For example, for the triple (P, R, S) we have the following result, obtained in this way.

Proposition 5. *For given P and R, the extremal values of E are attained at two isosceles triangles with the length a of equal sides, satisfying the following equation:*

$$a^4 - 4P^2Ra + P^2R^2 = 0.$$

The length of base in each case is equal to $a^2R^{-1}\sqrt{4R^2 - a^2}$.

The proof is based on the following analog of Proposition 3 which can be proven, using the method of Lagrange multipliers.

Proposition 6. *For given P and R, the extrema of S are attained at two isosceles triangles with circumradius R and perimeter P.*

An interesting problem is to obtain precise description of the range of symmetric coordinates for each of the four triples given in Theorem 5. Analogous constructions and problems can be considered for n-gons with $n \geq 4$. Some results in this direction based on consideration of dual extremal problems are given in [4].

REFERENCES

1. Alaoui, A., Zeggar, A. Area and perimeter foliations on spaces of polygons. *Graduate J. Math.*, **4**, 1 (2019), 18-29.

2. Berger, M. Geometry. *Springer*, 2007.

3. Giorgadze, G., Khimshiashvili, G. On area foliation in spaces of triangles. *Bull. Georg. Natl. Acad. Sci.*, **14**, 2 (2020), 7-13.

4. Khimshiashvili, G. Isoperimetric duality in polygon spaces. *Bull. Georg. Natl. Acad. Sci.*, **14**, 1 (2020), 18-22.

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