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DUAL EXTREMAL PROBLEMS IN POLYGON SPACES

Giorgi Khimshiashvili

Abstract. We discuss dual extremal problems in configuration spaces of planar polygons and present some topological information on the level sets of functions considered.

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1 Introduction. The aim of this paper is to present several results on the topology of level surfaces of the oriented area, obtained by applying the general approach to dual extremal problems, developed in [1] to configuration spaces of planar polygons, studied in [2], [3]. Namely, we use the duality between extremal problems to dualize the results on the isoperimetric problem in the space of planar polygons, presented in [2], [3]. This approach enables us to describe the homotopy structure of level surfaces of the oriented area and electrostatic energy in configurations space of planar polygons.

2 Extremal problems in planar polygon spaces. Let $n \geq 3$ be a natural number. As was shown in [3] the configuration space (aca moduli space) $M_1(n)$ of oriented planar n-gons with perimeter equal to 1 is naturally diffeomorphic to the complex projective space $\mathbb{CP}^{n-2}$. The configuration space $C(n)$ of all non-degenerate planar n-gons is naturally diffeomorphic to the open cone over $M_1(n)$. Thus $\dim C(n) = 2n - 3$.

We are concerned with the topology of levels sets of certain differential functions, defined on open dense subsets of $C(n)$. For an oriented planar n-gon X and non-zero integer k, let $P(X)$ be the perimeter of X, $A(X)$ the oriented area of X, and $S_k(X)$ the sum of k-th powers of lengths of the sides and diagonals of X. Notice that S_{-1} coincides with the electrostatic energy E of unit charges with Coulomb interaction placed at the vertices of X. For the real-valued function f on a subset U of $C(n)$ and real number a, we denote by f_a the level set $\{f = a\}$ and call it f-level. Regular n-gon is defined as a convex equilateral and equiangular n-gon. For $n \geq 5$, regular star n-gon is defined as a self-intersecting equilateral equiangular n-gon. For an oriented regular n-gon X, one has the well-defined *winding number* $w(X)$ equal to the winding number of its oriented boundary about the center of circumscribed circle. This number lies in the interval $[-[n-1/2], [n-1/2]]$, where the inner square brackets denote the integer part, and determines the shape of regular n-gon up to a homothety. Obviously, the winding number of convex regular n-gon with positive (negative) orientation equals 1 (-1). For even n, we also consider the degenerate regular n-gon having only two different vertices and call it the *complete fold*.

We consider several pairs of dual extremal problems in $C(n)$ in the sense of [1]. Namely, we fix the value a of one of the above functions f and search for the critical points of restriction $g|f_a$ of another function g. In other words, this is the constrained extremal problem with target function g and constraint f, which we denote $\mathbb{P}(g|f)$. Its dual problem $\mathbb{P}^*$ is defined as the constrained extremal problem $\mathbb{P}(f|g)$ with target function f and constraint g. From Lagrange rule follows that the solutions to both dual problems coincide with the singular set $S(\Phi)$ of mapping $\Phi = (g, f)$. As was shown in [1], if $p \in S(\Phi)$ is a non-degenerate critical point of the problem $\mathbb{P}(g|f)$ then p is also a non-degenerate critical point of the dual problem $\mathbb{P}(f|g)$ and their Morse indices are related as follows. Namely, if the gradients $\nabla f, \nabla g$ at point p are codirected then the sum of two indices is equal to the dimension of an ambient manifold and these gradients are oppositely directed then the two indices coincide. The results given in [1] are in the spirit the so-called *Morse theory for several functions* developed by S.Smale and develop some aspects of the theory presented in [4].

In this paper, we are concerned with the topology of levels of oriented area A and Coulomb energy in $C(n)$. The main results are derived by comparing the problems $\mathbb{P}(A|P)$ and $\mathbb{P}(E|P)$ with their dual problems. If the perimeter of n-gon is chosen as constraint we speak of an *isoperimetric problem.* In particular, the problem $\mathbb{P}(A|P)$ in $C(n)$ is referred to as the *classical isoperimetric problem* for n-gons. The following result proven in [3] plays an important role in the sequel.

Proposition 1. *(see [3]) The critical points of the problem $\mathbb{P}(A|P)$ in $C(n)$ are exactly the (representatives of) regular n-gon with both orientations and regular star n-gons with both orientations, and also the complete fold if n is even. All these critical points are non-degenerate. The Morse index of A at an oriented regular n-gon X with winding number $w(X)$ equals $2w(X)-2$ if $w(X)$ is negative, and equals $2n-2w(X)-2$ if $w(X)$ is positive. For even n, the Morse index of area at the complete fold is $n-2$.*

Notice that a non-zero level surface of P in $C(n)$ is naturally diffeomorphic to $M_1(n)$. Hence this result shows that A is a Morse function on $M_1(n)$ with exactly one critical point of each even index $0, 2, \ldots, 2n$. Notice that this fits the fact that $M_1(n)$ is diffeomorphic to $\mathbb{CP}^n$. In fact, the oriented area is a perfect Morse function on $M_1(n)$. In particular, the positively (negatively) oriented regular n-gon is the point of maximum (minimum) of A. For $n = 5$, one has only two regular stars with winding numbers ± 2 and they are saddles of area with Morse indices 2 and 4 respectively. Comparing the directions of gradients of P and A at a regular n-star and using Proposition 1 we obtain the following result which is in a sense dual to Proposition 1.

Proposition 2. *Let A_a be a non-zero level set of the area in $C(N)$. If $a > 0$ then the critical points of perimeter restricted to A_a are exactly the regular n-stars with positive winding number. Each such critical point X is non-degenerate and the Morse index of perimeter at X equals $2w(X) - 2$. If $a < 0$ then the critical points of perimeter restricted to A_a are exactly the regular n-stars with negative winding number. Each such critical point X is non-degenerate and the Morse index of perimeter at X equals $2n - 2w(X) - 2$.*

Combining this result with standard methods of Morse theory we obtain the results on topology of area levels presented in the next section.

3 Homotopy structure of area levels.

First of all, for $n = 3$ and $n = 4$ there are no regular n-stars so there is only one critical point of P in a non-zero level A_a and by Morse theory the gradient flow of P defines a contracting homotopy of A_a.

Theorem 1. *For $n = 3, 4$, non-zero levels of area in $C(n)$ are contractible non-compact smooth submanifolds of $C(n)$ of dimension $2n - 4$.*

For $n \geq 5$, using the cell decomposition provided by Morse theory it is easy to describe the homotopy type of non-zero level A_a.

Theorem 2. *For $n \geq 5$, a non-zero level A_a in $C(n)$ is homotopy equivalent to a CW-complex having exactly one cell in each even dimension $k = 0, 2, \ldots, 2n - 4$.*

In particular, the non-zero area levels are non-contractible for $n \geq 5$ and have Euler characteristics equal to $n - 1$. One can further explicate the above results applying more elaborate methods of Morse theory but we do not pursue this possibility here for the reason of space.

It is worth noting that the approach and reasoning presented above enable one to obtain analogous results for functions S_r. As an illustration we present two results concerned with the electrostatic (Coulomb) energy. Their proofs are based on the interpretation of regular stars as critical points of Coulomb energy in isoperimetric setting presented in [2].

Theorem 3. *For $n = 3, 4$ and $a > 0$, any Coulomb level E_a in $C(n)$ is a contractible non-compact smooth submanifold of $C(n)$ of dimension $2n - 4$.*

Theorem 4. *For $n \geq 5$ and $a > 0$, any Coulomb level E_a in $C(n)$ is homotopy equivalent to a CW-complex having exactly one cell in each even dimension $k = 0, 2, \ldots, 2n - 4$.*

Analogous results can be obtained for level surfaces of A and E in configuration spaces of cyclic n-gons. Further results in this direction are given in [1].

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Author(s) address(es):

Giorgi Khimshiashvili
Institute for Fundamental and Interdisciplinary Mathematical Research
Ilia State University
K. Cholokashvili Ave. 3/5, 0162 Tbilisi, Georgia
E-mail: giorgi.khimshiashvili@iliauni.edu.ge