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INVESTIGATION AND APPROXIMATE SOLUTION OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATION OF DIFFUSION TYPE

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Abstract. Investigation and numerical solution of the nonlinear integro-differential equation of parabolic type is considered. Integro-differential models of this type are based on the system of Maxwell equations and appear in various diffusion problems. Unique solvability, asymptotic behavior of the solution of the initial-boundary value problem and convergence of the finite-difference scheme are given.

Keywords and phrases: Diffusion problem, Maxwell system, nonlinear integro-differential equation, unique solvability, asymptotic behavior, finite-difference scheme, convergence.

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The nonlinear integro-differential equations describe various processes in physics, economics, technology and so on (see, for example, [1]-[7] and references therein). The study of qualitative and structural properties of the solutions of initial-boundary problems for these models, construction and investigation of discrete analogues are very important.

One type of integro-differential nonlinear parabolic model has the following form

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[A(S)\frac{\partial U}{\partial x}\right] = F\left(x,t\right), \tag{1}$$

where

$$S(x,t) = \int_0^t U^2(x,\tau)d\tau, \tag{2}$$

or

$$S(x,t) = \int_0^t \left(\frac{\partial U(x,\tau)}{\partial x}\right)^2 d\tau. \tag{3}$$

Here $A = A(S)$, $F = F(x,t)$ are given functions and $U = U(x,t)$ is an unknown.

Models of (1), (3) kind are obtained at the mathematical simulation of the diffusion processes of electromagnetic field penetration into a substance. Based on the Maxwell system [8] this model at first appeared in [9]. Many other processes are described by the integro-differential system obtained in [9] (see, for example, [1], [2]). Lots of scientific works are dedicated to the investigation and the numerical resolution of the initial-boundary value problems for these types of models (see, for example, [1]-[3], [9]-[19] and references therein). The existence, uniqueness and asymptotic behavior of the solution for such type equations and systems are studied in the works [1]-[3], [9]-[14], [16], [17] and in a number of other works as well (for more detail citations see, for example, [1], [2]).

For the first time, models of (1), (2) kind are obtained in [3] as a mathematical generalization of (1), (3) type models.

The principal characteristic peculiarity of the equations (1), (2) and (1), (3) is connected with the appearance of the nonlinear time integral term in the coefficient with derivative.

The present work is dedicated to the investigation and approximate resolution of the initial-boundary value problem for one generalization of the models (1), (2) and (1), (3). In the rectangle $Q = (0,\,1) \times (0,\,T]$ we consider the following problem:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left\{\left[1 + \int_0^t \left(U^2 + \left(\frac{\partial U}{\partial x}\right)^2\right) d\tau\right] \frac{\partial U}{\partial x}\right\} = F\,(x,t)\,, \tag{4}$$
$$U(0,\,t) = U(1,\,t) = 0, \quad U(x,\,0) = U_0(x),$$

where $U_0 = U_0(x)$ is given the function.

Using one modification of compactness method, developing in [20] (see also [2], [3], [5], [9]-[11]) the following uniqueness and existence statement takes place.

Theorem 1. *If $F \in W_2^1(Q)$, $F(x,0) = 0$, $U_0 \in \overset{o}{W}{}_2^1(0,1)$, then there exists a unique solution $U = U(x,t)$ of the problem* (4) *satisfying the following properties:*

$$U \in L_4(0,T;\overset{o}{W}{}_4^1(0,1)), \quad \frac{\partial U}{\partial t} \in L_2(Q), \quad \frac{\partial^2 U}{\partial x^2}, \quad \frac{\partial^2 U}{\partial t \partial x} \in L_2(Q).$$

Here usual well-known spaces are used. The proof of the formulated Theorem 1 is divided into several steps applying the Galerkin method and the method of compactness.

It is easy to deduce exponential stabilization as $t \to \infty$ of the solution of problem (4) in the norm of the space $L_2(0,1)$. The analogical results for the first derivatives are true.

Theorem 2. *If $F(x,t) \equiv 0$, $U_0 \in W_2^2(0,1)$, $U_0(0) = U_0(1) = 0$, then for the solution $U = U(x,t)$ of the problem* (4) *the following estimate is true*

$$\left\|\frac{\partial U}{\partial t}\right\| + \left\|\frac{\partial U}{\partial x}\right\| \le C \exp\left(-\frac{t}{2}\right).$$

Here C denotes the positive constant independent from time variable t.

In order to describe the space-discretization for problem (4), the following grid is introduced $\bar{\omega}_h = \{x_i = ih,\ \ i = 0,1,...,M\}$, with $h = 1/M$. The semi-discrete approximation at (x_i,t) is designed by $u_i = u_i(t)$. The exact solution of the problem (4) at (x_i,t), denoted by $U_i = U_i(t)$, is assumed to exist and be smooth enough.

Approximating the space derivatives by a forward and backward differences:

$$u_{x,i} = \frac{u_{i+1} - u_i}{h}, \quad u_{\bar{x},i} = \frac{u_i - u_{i-1}}{h},$$

let us correspond the following semi-discrete scheme to the problem (4):

$$\frac{du_i}{dt} - \left\{\left[1 + \int_0^t (u_i^2 + u_{\bar{x},i}^2) d\tau\right] u_{\bar{x},i}\right\}_{x,i} = f_i, \quad i = 1,2,...,M-1, \tag{5}$$
$$u_0\,(t) = u_M\,(t) = 0, \quad u_i\,(0) = U_{0,i}, \quad i = 0,1,...,M.$$

The semi-discrete scheme (5) is stable with respect to the initial data and the right-hand side in the norm

$$\|u\|_h = (u,u)_h^{1/2}, \quad (u,v)_h = \sum_{i=1}^{M-1} u_i v_i h.$$

Theorem 3. *If the problem* (4) *has a sufficiently smooth solution* $U = U(x,t)$, *then the solution* $u(t) = (u_1(t), u_2(t), ..., u_{M-1}(t))$ *of the semi-discrete scheme* (5) *tends to* $U(t) = (U_1(t), U_2(t), ..., U_{M-1}(t))$ *as* $h \to 0$ *and the following estimate is true*

$$\|u(t) - U(t)\|_h \le Ch.$$

Here C denotes the positive constant independent from spatial step h.

In order to describe the time-discretization for problem (4), let us introduce the net $\omega_\tau = \{t_j = j\tau,\ j = 0,1,...,J\}$, with $\tau = T/J$ and $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau$, $u_i^j = u(x_i, t_j)$.

On $\overline{\omega}_{h\tau}$ let us correspond to the problem (4) the following finite difference scheme:

$$\begin{gathered} \frac{u_i^{j+1} - u_i^j}{\tau} - \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_i^k)^2 + \left(u_{\bar{x},i}^k \right)^2 \right] \right) u_{\bar{x},i}^{j+1} \right\}_{x,i} = f_i^{j+1}, \\ i = 1,2,...,M-1; \quad j = 0,1,...,J-1, \\ u_0^j = u_M^j = 0, \quad j = 0,1,...,J, \quad u_i^0 = U_{0,i}, \quad i = 0,1,...,M. \end{gathered} \tag{6}$$

It is proved that system (6) is uniquely solvable and stability is also studied.

Theorem 4. *If the problem* (4) *has a sufficiently smooth solution* $U = U(x,t)$, *then the solution* $u^j = \left(u_1^j, u_2^j, ..., u_{M-1}^j\right)$ *of the difference scheme* (6) *tends to the* $U^j = \left(U_1^j, U_2^j, ..., U_{M-1}^j\right)$ *as* $\tau \to 0$, $h \to 0$ *and the following estimate is true*

$$\left\| u^j - U^j \right\|_h \le C(\tau + h), \quad j = 1,2,\dots,J.$$

Here C is positive constant independent from time and spatial steps τ and h.

Note that for solving the difference scheme (6) Newton iterative process is used. Various numerical experiments are done which agree with theoretical research.

REFERENCES

1. Jangveladze, T., Kiguradze Z., Neta B. Numerical Solution of Three Classes of Nonlinear Parabolic Integro-Differential Equations. 2016, ACADEMIC PRESS, ISBN: 978-0-12-804628-9. Elsevier/Academic Press, Amsterdam, 2015.
2. Jangveladze, T. Investigation and Numerical Solution of Nonlinear Partial Differential and Integro-Differential Models Based on System of Maxwell Equations. *Mem. Differential Equations Math. Phys.*, **76**, (2019).
3. Dzhangveladze, T. An investigation of the First Boundary-Value Problem for Some Nonlinear Parabolic Integrodifferential Equations (Russian). *Tbilisi State University*, 1983.
4. Volterra, V. Theory of Functionals and Integral and of Integral and Integro-Differential Equations *Blackie & Son Ltd, London*, 1931.

5. Lions, J.-L. Quelques Methodes de Resolution des Problemes aux Limites Non-lineaires. *Dunod Gauthier-Villars, Paris*, 1969.
6. Gripenberg, G., Londen, S., Staffans, O. Volterra Integral and Functional Equations. *Cambridge University Press, Cambridge*, 1990.
7. Lakshmikantham, V., Rao, M. Theory of Integro-Differential Equations. *CRC Press*, 1995.
8. Landau, L.D., Lifshitz, E.M. Electrodynamics of Continuous Media (Russian). *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow*, 1957.
9. Gordeziani, D., Dzhangveladze, T., Korshia, T. Existence and uniqueness of a solution of certain nonlinear parabolic problems (Russian). *Differ. Uravn.*, **19**, (1983), 1197-1207; English translation: *Differ. Equ.*, **19**, (1983), 887-895.
10. Dzhangveladze, T. First boundary value problem for a nonlinear equation of parabolic type (Russian). *Dokl. Akad. Nauk SSSR*, **269**, (1983), 839-842. English translation: *Soviet Phys. Dokl.*, **28**, (1983), 323-324.
11. Laptev, G. Quasilinear parabolic equations which contains in coeffcients Volterra's operator (Russian). *Math. Sbornik*, **136**, (1988), 530-545. English translation: *Sbornik Math.*, **64**, (1989), 527-542.
12. Lin, Y., Yin, H. Nonlinear parabolic equations with nonlinear functionals. *J. Math. Anal. Appl.*, **168**, (1992), 28-41.
13. Long, N., Dinh, A. Nonlinear parabolic problem associated with the penetration of a magnetic field into a substance. *Math. Meth. Appl. Sci.*, **16**, (1993), 281-295.
14. Jangveladze, T. On one class of nonlinear integro-differential equations. *Sem. I.Vekua Inst. Appl. Math., REPORTS*, **23**, (1997), 51-87.
15. Jangveladze, T. Convergence of a difference scheme for a nonlinear integro-differential equation. *Proc. I. Vekua Inst. Appl. Math.*, **48**, (1998), 38-43.
16. Kiguradze, Z. The asymptotic behavior of the solutions of one nonlinear integro-differential model. *Sem. I.Vekua Inst. Appl. Math., REPORTS*, **30**, (2004), 21-32.
17. Jangveladze, T., Kiguradze, Z. On the stabilization of solutions of an initial-boundary value problem for a nonlinear integro-differential equation (Russian). *Differ. Uravn.*, **43**, (2007), 833-840. English translation: *Differ. Equ.*, **43**, (2007), 854-861.
18. Liao, H., Zhao, Y. Linearly localized difference schemes for the nonlinear Maxwell model of a magnetic field into a substance. *Appl. Math. Comput.*, **233**, (2014), 608-622.
19. Sharma, N., Khebchareon M., Sharma K., Pani A. Finite element Galerkin approximations to a class of nonlinear and nonlocal parabolic problems. *Numer. Meth. Part. Diff. Eq.*, **32**, (2016), 1232-1264.
20. Vishik, M.I. Solvability of boundary-value problems for quasi-linear parabolic equations of higher orders (Russian). *Mat. Sb. (N.S.)*, **59 (101)**, (1962), 289-325.
21. Samarskii, A.A. The Theory of Difference Schemes (Russian). *Moscow*, 1977.

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