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THE BOUNDARY VALUE PROBLEMS FOR PIECEWISE-HOMOGENEOUS VISCOELASTIC PLATE

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Abstract. A piecewise-homogeneous viscoelastic plate with a finite cut, which intersect the interface at a rigid angle is considered. The complex potentials are represented with jumps of stresses and displacements. The first main boundary value problem of elasticity theory, when the crack boundary is loaded with symmetrical forces, is considered. The problem is reduced to the system of singular integral equations with respect to the crack opening. The asymptotic estimates are obtained.

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1 We will consider a piecewise-homogeneous viscoelastic plate (v_k, E_k) with a finite cut, that crosses the interface at right angles. The plate consists of two half-planes of dissimilar materials

$$S^{(1)} = \{z|\ \mathrm{Re} z > 0,\ z \notin l_1 = [0,b]\},\ \ S^{(2)} = \{z|\mathrm{Re} z < 0,\ z \notin l_2 = (-a,0)\},\ (a,b) > 0.$$

On the boundary of the cut the jumps of the stresses and displacement are obtained:

$$\begin{aligned} &\sigma_y^{(k)+} - \sigma_y^{(k)-} = f_1^{(k)}(x,t),\ \ \tau_{xy}^{(k)+} - \tau_{xy}^{(k)-} = f_2^{(k)}(x,t), \\ &u_k^+ - u_k^- = f_3^{(k)}(x,t),\ \ v_k^+ - v_k^- = f_4^{(k)}(x,t),\ x \in l_k,\ k = 1,2. \end{aligned} \tag{1}$$

At the interface of the two materials we have the continuity conditions:

$$\sigma_x^{(1)} = \sigma_x^{(2)},\ \tau_{xy}^{(1)} = \tau_{xy}^{(2)},\ \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y},\ \frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial y}. \tag{2}$$

Using the known relation by a solving of the problems of linear conjugation [1] the complex potentials are represented as follows

$$\Phi_k(z,t) = \frac{1}{2\pi i}\int_{l_k} \frac{a_k(x,t)dx}{x-z} + W_k(z,t),\ \ \Psi_k(z,t) = \frac{1}{2\pi i}\int_{l_k} \frac{b_k(x,t)dx}{x-z} + Q_k(z,t), \tag{3}$$

where

$$a_k(x,t) = \frac{2\mu_k}{\varkappa_k + 1}\left[\frac{f_1^{(k)} - i f_2^{(k)}}{2\mu_k} + (I-L)_{(k)}^{-1}(f_3^{'(k)} + f_4^{'(k)})\right],$$

$$b(x,t) = f_1^{(k)} + i f_2^{(k)} - \frac{2\mu_k}{\varkappa_k + 1}\left[a_k(x,t) + \overline{a_k(x,t)} + x a_k'(x,t)\right],$$

where t is time, elasto-instal deformation modulus are $E_k(t) = E_k = const$, the lateral contraction coefficients for elastic deformation $v_k(t)$ and creep deformation $v_k(t,\tau)$ are the some and constant: $v_k(t) = v_k(t,\tau) = const$,

$$(I-L)_{(k)}g(t) = g(t) - \int_{t_0}^{t} E_k \frac{\partial}{\partial\tau} C_k(t,\tau) g(\tau) d\tau, \ 2\mu_k = \frac{E_k}{1+v_k}, \ \varkappa_k = \begin{cases} 3-4v_k, \\ \dfrac{3-v_k}{1+v_k}, \end{cases} \quad k=1,2,$$

$C_k(t,\tau)$ are creep measure of the materials, τ_0 is material age at the time of loading. $W_k(z,t)$, $Q_k(z,t)$ are unknown analytic functions in the half-planes $S^{(k)} + l_k$ respectively, which will be defined from condition (2).

From conditions (2)–(3), using the Cauchy Theorem the known relations gives us the system of algebraic equations, whose solutions are:

$$\begin{aligned} W_1(z,t) &= e_1\bar\eta_1(-z,t) + e_1\bar A_1(-z,t) + h_1 A_2(z,t), \\ \Omega_1(z,t) &= -e_1\bar\eta_1(-z,t) + m_1\bar A_1(-z,t) + m_2 A_2(z,t) + h_2\eta_2(z,t), \\ W_2(z,t) &= h_3 A_1(z,t) - e_2\bar A_2(-z,t) - e_2\bar\eta_2(-z,t), \\ \Omega_2(z,t) &= h_4\eta_1(z,t) - m_1 A_1(z,t) - m_2\bar A_2(-z,t) + e_2\bar\eta_2(-z,t), \end{aligned} \tag{4}$$

where $\Omega_k(z,t) = zW_k'(z,t) - Q_k(z,t)$, $\eta_k(z,t) = zA_k'(z,t) - B_k(z,t)$, $e_k = \frac{\mu_2-\mu_1}{\varkappa_k\mu_{3-k}+\mu_k}$, $m_1 = h_2 - h_4$, $m_2 = h_3 - h_1$, $h_k = \frac{(\varkappa_{3-k}+1)\mu_k}{\varkappa_k\mu_{3-k}+\mu_k}$, $h_{k+2} = \frac{(\varkappa_{3-k}+1)\mu_k}{\varkappa_{3-k}\mu_k+\mu_{3-k}}$.

2 First main boundary value problem of the theory of viscoelasticity. Now we consider the first main boundary-value problem of the theory of elasticity, when the crack boundary is loaded with symmetrical forces, then the boundary conditions have the form:

$$\begin{aligned} &\sigma_y^{(k)+} = \sigma_y^{(k)-} = N_k(x,t), \quad \tau_{xy}^{(k)+} = \tau_{xy}^{(k)-} = 0, \\ &u_k^+ - u_k^- = 0, \quad v_k^+ - v_k^- = v_k(x,t), \quad k=1,2 \end{aligned} \tag{5}$$

where $N_k(x,t)$ are given continuous functions, $v_k(x,t)$ are unknown functions, which depict the opening of a crack at the corresponding points. By considering the equations

$$a_k(x,t) = 2it_k f_k(x,t), \quad b_k(x,t) = -2it_k x f_k'(x,t), \quad f_k(x,t) = (I-L)_k^{-1} v_k'(x,t), \quad t_k = \frac{\mu_k}{\varkappa_k+1}$$

from (4) we have

$$\begin{aligned} \pi W_k(z,t) &= (-1)^k e_k t_k \int_{l_k} \tfrac{x f_k'(x,t)dx}{x+z} + (-1)^k e_k t_k \int_{l_k} \tfrac{x f_k(x,t)dx}{(x+z)^2} + h_k t_{3-k} \int_{l_{3-k}} \tfrac{f_{3-k}(x,t)dx}{x-z}, \\ \pi Q_k(z,t) &= e_k t_k \int_{l_k} \tfrac{x f_k'(x,t)dx}{x+z} + (-1)^k m_k t_k \int_{l_k} \tfrac{f_k(x,t)dt}{x+z} + 2(-1)^k e_k t_k z \int_{l_k} \tfrac{(x f_k(x,t))'dx}{(x+z)^2} \\ &\quad - h_{k+2} t_k \int_{l_{3-k}} \tfrac{x f_{3-k}'(x,t)dx}{x-z} + (-1)^k m_{3-k} t_{3-k} \int_{l_{3-k}} \tfrac{x f_{3-k}'(x,t)dt}{(x-z)^2}. \end{aligned} \tag{6}$$

From (5), (6) we obtain a system of singular integral equations with fixed singularity:

$$\frac{1}{\pi}\int_0^1\left\{\frac{1}{x-s}-\frac{2e_k+m_k}{2(x+s)}-\frac{2e_kx}{(x+s)^2}+\frac{4e_kx^2}{(x+s)^3}\right\}f_k((-1)^{k+1}x,t)dx$$
$$+\frac{1}{\pi}\int_0^1\left\{\frac{-(h_k+h_{k+2})t_{3-k}}{t_k(x+s)}+\frac{m_{3-k}t_{3-k}x}{t_k(x+s)^2}\right\}f_{3-k}((-1)^{k+1}x,t)dx=N_k((-1)^{k+1}s,t), \quad (7)$$
$$s>0,\ k=1,2.$$

3 Asymptotic study. Find the behavior of solutions of system (7) in the singular points, for this present the solution in the neighborhood of the point $s=1$ as follows

$$f_1(s,t)=(1-s)^{-\alpha}g_1(s,t),\ \ f_2(-x,t)\equiv\varphi(s,t)=(1-s)^{-\delta}\varphi_1(s,t),\ \ 0<\alpha,\ \delta<1$$

$$J_k(s,t)=\frac{1}{\pi}\int_0^1\left\{\frac{1}{x-s}-\frac{2e_k+m_k}{2(x+s)}-\frac{2e_kx}{(x+s)^2}+\frac{4e_kx^2}{(x+s)^3}\right\}f_k((-1)^{k+1}x,t)dx$$
$$-\frac{1}{\pi}\int_0^1\left\{\frac{(h_k+h_{k+2})t_{3-k}}{t_k(x+s)}-\frac{m_{3-k}t_{3-k}x}{t_k(x+s)^2}\right\}f_{3-k}((-1)^{k+1}x,t)dx$$

are regular functions in the point $s=1$, $J_k(s)=J_k(1)+J_k'(1)(1-s)+O((1-s)^2),\ k=1,2.$

According to the well-known theorems [2], about the behavior of Cauchy type integral in the ends of the integration line:

$$-\mathrm{ctg}\pi\alpha g_1(1,t)(1-s)^{-\alpha}+O((1-s)^{-\alpha_0})+J_1(1,t)+\cdots=N_1(1,t),$$

$$-\mathrm{ctg}\pi\delta\varphi_1(1,t)(1-s)^{-\delta}+O((1-s)^{-\delta_0})+J_2(1,t)+\cdots=N_2(-1,t),$$

$$\alpha_0<\alpha,\ \ \delta_0<\delta.$$

Whence there follows $\mathrm{ctg}\pi\alpha=0$, $\mathrm{ctg}\pi\delta=0$ and $\alpha=\beta=\frac{1}{2}$.

Now present the solution of system (7) in the neighborhood of the point $s=0$ as follows

$$f_1(s,t)=s^{-\beta}g_2(s,t),\ \ \varphi(s,t)=s^{-\gamma}\varphi_2(s,t),\ \ 0<\beta,\ \gamma<1,$$

$$J_{k+2}(s)=\frac{1}{\pi}\int_0^1\left\{-\frac{2e_kx}{(x+s)^2)}+\frac{4e_kx^2}{(x+s)^3}\right\}f_k((-1)^{k+1}x,t)dx$$
$$+\frac{1}{\pi}\int_0^1\left\{\frac{m_{3-k}t_{3-k}x}{t_k(x+s)^2}\right\}f_{3-k}((-1)^{k+1}x,t)dx,\ \ k=1,2$$

are regular functions in the point $s=0$. Since

$$\frac{1}{\pi}\int_0^1\frac{f_1(x,t)dx}{x-s}=\mathrm{ctg}\pi\beta g_2(0,t)s^{-\beta}+O(s^{-\beta_0})+O(s),$$

$$\frac{1}{\pi}\int_0^1\frac{\varphi(x,t)dx}{x-s}=\mathrm{ctg}\pi\gamma\varphi_2(0,t)s^{-\gamma}+O(s^{-\gamma_0})+O(s),\ s\to0,\ \beta_0<\beta,\ \gamma_0<\gamma,$$

from (7) we have

$$\begin{aligned}
&-\mathrm{ctg}\pi\beta g_2(0,t)s^{-\beta}+O(s^{-\beta_0})-A_1\tfrac{g_2(0,t)}{\sin\pi\beta}s^{-\beta}+A_1O(s^{-\beta_0})+A_1O(s)\\
&-D_1\tfrac{\varphi_2(0,t)}{\sin\pi\gamma}s^{-\gamma}+D_1O(s^{-\gamma_0})+D_1O(s)=N_1(0,t),\\
&-\mathrm{ctg}\pi\gamma\varphi_2(0,t)s^{-\gamma}+O(s^{-\gamma_0})-A_2\tfrac{\varphi_2(0,t)}{\sin\pi\gamma}s^{-\gamma}+A_2O(s^{-\gamma_0})+A_2O(s)\\
&-D_2\tfrac{g_2(0,t)}{\sin\pi\beta}s^{-\beta}+D_2O(s^{-\beta_0})+D_2O(s)=N_2(0,t),
\end{aligned}\tag{8}$$

where $A_k=\frac{2e_k+m_k}{2}$, $D_k=\frac{(h_k+h_{k+2})t_{3-k}}{t_k}$.

We can show that $\beta=\gamma$, therefore from (8) it follows that

$$\mathrm{ctg}\pi\beta+\frac{A_1}{\sin\pi\beta}+\frac{D_1}{\sin\pi\beta}\frac{\varphi_2(0,t)}{g_2(0,t)}=0,\ \ \mathrm{ctg}\pi\beta+\frac{A_2}{\sin\pi\beta}+\frac{D_2}{\sin\pi\beta}\frac{g_2(0,t)}{\varphi_2(0,t)}=0.$$

For satisfaction both of them must be performed the following conditions:

$$A_1+D_1\frac{\varphi_2(0,t)}{g_2(0,t)}=A_2+D_2\frac{g_2(0,t)}{\varphi_2(0,t)}\equiv\lambda,\ \ (A_1-A_2)^2+4D_1D_2\geq 0,\ |\lambda|\leq 1.$$

Then we get

$$\beta=\gamma=\frac{1}{\pi}\arccos\lambda.$$

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