Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 34, 2020

THE SOLUTION OF SOME PROBLEMS OF THE THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR A CIRCULAR RING

Bakur Gulua　　Tengiz Meunargia

Abstract. In this work we consider the two-dimensional version of statics of the linear theory of elastic materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures. Some BVPs are solved for a circular ring.

Keywords and phrases: Thermoelasticity, microtemperature, boundary value problem.

AMS subject classification (2010): 74F05, 74G10.

1 Basic equations.

The basic system of equations of the theory of thermoelasticity with microtemperatures can be written in the form [1, 2]:

$$\begin{aligned} &\mu\Delta u_\alpha + (\lambda+\mu)\partial_\alpha\theta - \beta\partial_\alpha T = 0,\\ &k_6\Delta w_\alpha + (k_4+k_5)\partial_\alpha\Theta - k_3\partial_\alpha T - k_2 w_\alpha = 0,\\ &k\Delta T + k_1\vartheta = 0, \end{aligned} \tag{1}$$

where $\lambda,\ \mu,\ \beta,\ k,\ k_1,...,k_6$ are constitutive coefficients, u_α are components of the displacement vector $\mathbf{u}$ of the point (x_1,x_2), w_α are components of is the microtemperature vector $\mathbf{w}$ of the point (x_1,x_2), $T(x_1,x_2)$ is the temperature measured from the constant absolute temperature T_0 $(T_0>0)$, $\theta=\partial_1u_1+\partial_2u_2$, $\Theta=\partial_1w_1+\partial_2w_2$, $\Delta=\partial_{11}+\partial_{22}$ is the tow dimensional Laplace operator.

On the plane Ox_1x_2 a complex variable $z=x_1+ix_2$, where i the imaginary unit, and the following operators $\partial_z=0.5(\partial_1-i\partial_2)$, $\partial_{\bar z}=0.5(\partial_1+i\partial_2)$ are introduced. Then the system consisting of the equations (1) can be written in complex form as follows

$$\begin{aligned} &\mu\Delta u_+ + 2(\lambda+\mu)\partial_{\bar z}\theta - 2\beta\partial_{\bar z}T = 0,\\ &k_6\Delta w_+ + 2(k_4+k_5)\partial_{\bar z}\Theta - 2k_3\partial_{\bar z}T - k_2w_+ = 0,\\ &k\Delta T + k_1\vartheta = 0, \end{aligned} \tag{2}$$

where $\Delta=4\partial_z\partial_{\bar z};\ u_+:=u_1+iu_2;\ w_+:=w_1+iw_2$.

For the positive definiteness of the corresponding quadratic form will satisfy the conditions

$$k_4+k_5+k_6>0,\ \ k_2>0,\ \ k_1k_3-kk_2<0,\ \ k>0.$$

In [3] it is shown that the general solution of system (12) is represented as follows:

$$\begin{aligned} 2\mu u_+ &= \varkappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \\ &+\frac{\mu\beta}{\lambda+2\mu}\left\{\frac{k_2}{2k_3}\left[\varphi(z)+z\overline{\varphi'(z)}\right]-\frac{2k_1}{kk^*}\partial_{\bar z}\chi_1(z,\bar z)\right\}, \\ w_+ &= -\overline{\varphi''(z)} + \partial_{\bar z}[\chi_1(z,\bar z)+i\chi_2(z,\bar z)], \\ T &= \frac{k_2}{2k_3}[\varphi'(z)+\overline{\varphi'(z)}] - \frac{k_1}{2k}\chi_1(z,\bar z), \end{aligned} \tag{3}$$

where $\varkappa = \dfrac{\lambda+3\mu}{\lambda+\mu}$, $\varphi(z)$, $\phi(z)$ and $\psi(z)$ are the arbitrary analytic function of a complex variable z, $\chi_1(z,\bar z)$ is a general solution of the following Helmholtz equation $\Delta\chi_1 - k^*\chi_1 = 0$, $k^* = \dfrac{k_2 k - k_1 k_3}{k(k_4+k_5+k_6)} > 0$, $\chi_2(z,\bar z)$ is a general solution of the following Helmholtz equation $\Delta\chi_2 - \tilde k\chi_2 = 0$, $\tilde k = \dfrac{k_2}{k_6}$.

2 A problem for a circular ring. In this section, we solve a concrete boundary value problem for a concentric circular ring with radius R_1 and R_2 [4].

We consider the following problem

$$u_+ = \begin{cases} \sum\limits_{-\infty}^{+\infty} A'_n e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} A''_n e^{in\vartheta}, & |z| = R_2, \end{cases} \tag{4}$$

$$w_+ = \begin{cases} \sum\limits_{-\infty}^{+\infty} B'_n e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} B''_n e^{in\vartheta}, & |z| = R_2, \end{cases} \qquad T = \begin{cases} \sum\limits_{-\infty}^{+\infty} C'_n e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} C''_n e^{in\vartheta}, & |z| = R_2. \end{cases} \tag{5}$$

The analytic functions $\phi(z)$, $\psi(z)$, $\varphi(z)$ and the metaharmonic functions $\chi_1(z,\bar z)$, $\chi_2(z,\bar z)$ are represented as a series

$$\phi(z) = a\ln z + \sum_{-\infty}^{+\infty} a_n z^n, \quad \psi(z) = b + \sum_{-\infty}^{+\infty} b_n z^n, \quad \varphi'(z) = c\ln z + \sum_{-\infty}^{+\infty} c_n z^n, \tag{6}$$

$$\chi_\gamma(z,\bar z) = \sum_{-\infty}^{+\infty}\left(\alpha_{\gamma n} I_n\left(\sqrt{k^*}r\right) + \beta_{\gamma n} K_n\left(\sqrt{k^*}r\right)\right)e^{in\vartheta}, \tag{7}$$

where $I_n(\cdot)$ and $K_n(\cdot)$ are modified Bessel functions of order n.

In the boundary conditions (5) we substitute the corresponding expressions (6), (7) and compare the coefficients at identical degrees. We obtain the following system of equations

$$
\begin{gathered}
\frac{k_2}{k_3} c \ln R_\gamma+\frac{k_2}{2 k_3} c_0-\frac{k_1}{2 k}\left[\alpha_{10} I_0(\sqrt{k^*} R_\gamma)+\beta_{10} K_0(\sqrt{k^*} R_\gamma)\right]=C_{\gamma 0}, \\
-\frac{2 c}{R_\gamma}+\sqrt{k^*}\left[\alpha_{10} I_1(\sqrt{k^*} R_\gamma)-\beta_{10} K_1(\sqrt{k^*} R_\gamma)\right]=B_{\gamma 1}+\overline{B}_{\gamma 1}, \\
\sqrt{\tilde{k}}\left[\alpha_{20} I_1(\sqrt{\tilde{k}} R_\gamma)-\beta_{20} K_1(\sqrt{\tilde{k}} R_\gamma)\right]=-i(B_{\gamma 1}-\overline{B}_{\gamma 1}),
\end{gathered} \tag{8}
$$

$$
\begin{gathered}
\frac{k_2}{k_3}\left[R_\gamma^n c_n+R_\gamma^{-n} \bar{c}_{-n}\right]-\frac{k_1}{2 k}\left[\alpha_{1 n} I_n(\sqrt{k^*} R_\gamma)+\beta_{1 n} K_n(\sqrt{k^*} R_\gamma)\right]=C_{\gamma n}, \\
-n\left[R_\gamma^{n-1} c_n-R_\gamma^{-n-1} \bar{c}_{-n}\right] \\
+\frac{\sqrt{k^*}}{2}\Big[(I_{n-1}(\sqrt{k^*} R_\gamma)+I_{n+1}(\sqrt{k^*} R_\gamma)) \alpha_{1 n}-(K_{n-1}(\sqrt{k^*} R_\gamma)+K_{n+1}(\sqrt{k^*} R_\gamma)) \beta_{1 n}\Big] \\
-i \frac{\sqrt{\tilde{k}}}{2}\left[(I_{n-1}(\sqrt{\tilde{k}} R_\gamma)-I_{n+1}(\sqrt{\tilde{k}} R_\gamma) \alpha_{2 n})-(K_{n-1}(\sqrt{\tilde{k}} R_\gamma)-K_{n+1}(\sqrt{\tilde{k}} R_\gamma) \beta_{2 n})\right] \\
=B_{\gamma n+1}+\overline{B}_{\gamma 1-n}, \\
-n\left[R_\gamma^{n-1} c_n+R_\gamma^{-n-1} \bar{c}_{-n}\right] \\
+\frac{\sqrt{k^*}}{2}\Big[(I_{n-1}(\sqrt{k^*} R_\gamma)-I_{n+1}(\sqrt{k^*} R_\gamma)) \alpha_{1 n}-(K_{n-1}(\sqrt{k^*} R_\gamma)-K_{n+1}(\sqrt{k^*} R_\gamma)) \beta_{1 n}\Big] \\
-i \frac{\sqrt{\tilde{k}}}{2}\left[(I_{n-1}(\sqrt{\tilde{k}} R_\gamma)+I_{n+1}(\sqrt{\tilde{k}} R_\gamma) \alpha_{2 n})-(K_{n-1}(\sqrt{\tilde{k}} R_\gamma)+K_{n+1}(\sqrt{\tilde{k}} R_\gamma) \beta_{2 n})\right] \\
=-B_{\gamma n+1}+\overline{B}_{\gamma 1-n}, \quad \gamma=1,2.
\end{gathered} \tag{9}
$$

It is possible to show that the determinant of the systems (8) and (9) are other than zero. Equations (8), (9) uniquely determine all coefficients of decomposition of functions $\varphi(z)$, $\chi_1(z,\bar{z})$ and $\chi_2(z,\bar{z})$. Absolute and uniform convergence of the series obtained in the ring (including the contours) will be provided if the functions set on the boundary have sufficient smoothness.

In the boundary conditions (5) we substitute the corresponding expressions (6), (7):

$$
\begin{aligned}
&(\varkappa a+\bar{b}) i \vartheta+(\varkappa a-\bar{b}) \ln R_\gamma-\bar{a} e^{2 i \vartheta}+\sum_{-\infty}^{+\infty} \varkappa a_n R_\gamma^n e^{i n \vartheta} \\
&-\sum_{-\infty}^{+\infty}\left(n \bar{a}_n R_\gamma^n e^{-i(n-2) \vartheta}+\bar{b}_n R_\gamma^n e^{-i n \vartheta}\right)=E i \vartheta+\sum_{-\infty}^{+\infty} D_\gamma n e^{i n \vartheta}, \quad \gamma=1,2,
\end{aligned} \tag{10}
$$

where

$$
E=-\frac{\mu \beta}{\lambda+2 \mu} \frac{k_2}{2 k_3} c_{-1},
$$

$$
D_{\gamma 0}=-\frac{\mu \beta}{\lambda+2 \mu}\left\{\frac{k_2}{2 k_3}[\ln R_\gamma+1] c_{-1}-\frac{k_1}{k \sqrt{k^*}}[I_0(\sqrt{k^*} R_\gamma) \alpha_{1-1}-K_0(\sqrt{k^*} R_\gamma) \beta_{1-1}]\right\},
$$

$$
\begin{aligned}
D_{\gamma 1}= & -\frac{\mu \beta}{\lambda+2 \mu} \frac{k_2}{2 k_3}\left[c R_\gamma(2 \ln R_\gamma-1)+2 c_0 R_\gamma\right] \\
& +\frac{\mu \beta}{\lambda+2 \mu} \frac{k_1}{k \sqrt{k^*}}\left[I_1(\sqrt{k^*} R_\gamma) \alpha_{10}-K_1(\sqrt{k^*} R_\gamma) \beta_{10}\right],
\end{aligned}
$$

$$D_{\gamma n}=-\frac{\mu\beta}{\lambda+2\mu}\frac{k_2}{2k_3}\left[\frac{c_{n-1}}{n}R_\gamma^n+\bar{c}_{1-n}R_\gamma^{-n}\right]$$
$$+\frac{\mu\beta}{\lambda+2\mu}\frac{k_1}{k\sqrt{k^*}}\left[I_n(\sqrt{k^*}R_\gamma)\alpha_{1n-1}-K_n(\sqrt{k^*}R_\gamma)\beta_{1n-1}\right].$$

The requirement for uniqueness of displacements have the following form

$$\varkappa a+\bar{b}=E. \tag{11}$$

Comparison of terms involving $e^{in\vartheta}$ gives

$$\begin{aligned}&\varkappa\ln R_\gamma a-2R_\gamma^2\bar{a}_2+\varkappa a_0-\bar{b}_0=D_{\gamma 0},\\&-\bar{a}+\varkappa R_\gamma^2 a_2-R_\gamma^{-2}\bar{b}_{-2}=D_{\gamma 2},\\&\varkappa R_\gamma^n a_n+(n-2)R_\gamma^{-n+2}\bar{a}_{-n+2}-R_\gamma^{-n}\bar{b}_{-n}=D_{\gamma n},\quad \gamma=1,2.\end{aligned} \tag{12}$$

Equations (11), (12) uniquely determine all coefficients of decomposition of functions $\phi(z)$ and $\psi(z)$.

REFERENCES

1. Ieşan, D., Quintanilla, R. On a theory of thermoelasticity with microtemperatures. *J. Thermal Stresses*, **23** (2000), 199–215.
2. Scalia, A., Svanadze, M., Tracina, R. Basic theorems in the equilibrium theory of thermoelasticity with microtemperatures. *J. Thermal Stresses*, **33** (2010), 721–753.
3. Janjgava, R., Narmania, M. The solution of some two dimensional problems of thermoelasticity taking into account the microtemperature. *J. Thermal Stresses*, **39**, 1 (2016), 57–64.
4. Tsagareli, I., Svanadze, M. The solution of the stress problem of the theory of thermoelasticity with microtemperatures for a circular ring. *Seminar of I. Vekua Institute of Applied Mathematics, REPORTS*, **38** (2012), 62–68.

Received 30.05.2020; revised 22.07.2020; accepted 17.09.2020.

Author(s) address(es):

Bakur Gulua
I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia

Sokhumi State University
Politkovskaya str. 61, 0186 Tbilisi, Georgia
E-mail: bak.gulua@gmail.com

Tengiz Meunargia
I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
E-mail: tengizmeunargia37@gmail.com