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# ABSOLUTE CONVERGENCE OF GENERAL FOURIER SERIES

Leri Gogoladze Giorgi Cagareishvili

**Abstract**. The properties of the Fourier series of the functions from some differentiable class are well known for classical orthonormal systems (trigonometric, Haar, Walsh, etc.). On the other hand, S. Banach proved that good differential properties of a function do not guarantee the a.e. convergence of the Fourier series of this function with respect to general orthonormal systems(ONS). Therefore, in order to obtain well-known results for general ONS, we need to impose special conditions on the functions of the given system. In the present paper we find conditions on the functions of an ONS under which the special series of Fourier coefficients of functions from Lip1 are absolutely convergent.

**Keywords and phrases**: The Fourier series, general orthonormal systems, absolutely convergent.

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**1 Auxiliary notations and results.** Let $(\varphi_n)$ be an ONS on $[0,1]$, $(a_n) \in l_2$ is an arbitrary sequence, $p$ and $\gamma$ are arbitrary numbers $(1 \le p < 2)$. Fourier coefficients of $f \in L_2(0,1)$ function are

$$C_n(f) = \int_0^1 f(x)\varphi_n dx.$$

Suppose

$$Q_n(p,\gamma,a,t,x) = \sum_{k=1}^{n} k^\gamma |a_k|^{p-1} \varphi_k(x) r_k(t),$$

where $r_k$ are Rademacher functions (see [1, ch. 1]).

Let us introduce the basic notations

$$H_n(p,\gamma,a,t) = \frac{1}{n}\sum_{i=1}^{n-1}\left|\int_0^{\frac{i}{n}} Q_n(p,\gamma,a,t,x)dx\right|,$$

and

$$B_n(p,\gamma,a) = \left(\sum_{k=1}^{n} k^\gamma |a_k|^p\right)^{\frac{p-1}{p}}.$$

The following theorem holds

**Theorem A** (see [2]-[6]). *Let $(\varphi_n)$ be trigonometric, Haar or Walsh systems on $[0,1]$. Then for any $f \in Lip1$ holds*

$$\sum_{n=1}^{\infty} n^\gamma |C_n(f)|^p < +\infty,$$

*when* $1+\gamma-\frac{3}{2}p<0,\ 1<p<2.$

It is known (see [7]) that if $f\in L_2$ $(f\neq 0)$ is an arbitrary function and $(a_n)\in l_2$ is an arbitrary sequence of real numbers, then there exists an ONS $(\varphi_n)$ such that $C_n(f)=ba_n$, $n=1,2,...$ and $b$ is an absolute constant.

Now we suppose that, $h(x)=1$ when $x\in[0,1]$ and $a_n=n^{-\frac{3}{4}}$, $p=\frac{5}{3}$, $\gamma=1$, then as $(a_n)\in l_2$, there exists an ONS $(\varphi_n)$ such that $C_n(h)=bn^{-\frac{3}{4}}$.

Consequently

$$\sum_{n=1}^{\infty} n^{\gamma}|C_n(h)|^p=\sum_{n=1}^{\infty} n\cdot n^{-\frac{3}{4}\times\frac{5}{3}}b^{\frac{5}{3}}=b^{\frac{5}{3}}\sum_{n=1}^{\infty} n^{-\frac{1}{4}}=+\infty,$$

but $f\in Lip1$ and $1+\gamma-\frac{3}{2}p<0,\ 1<p<2.$

From here we conclude that Theorem A is not true for all ONS.

In this paper we find conditions on the functions ONS $(\varphi_n)$ under which

$$\sum_{n=1}^{\infty} n^{\gamma}|C_n(f)|^p<+\infty,\quad 1+\gamma-\frac{3}{2}p<0,\ \ 1<p<2$$

holds for any function $f\in Lip1$.

## 2 The main results.

**Theorem 1.** *Let* $(\varphi_n)$ *be an ONS on* $[0,1]$ *for which*

$$\sum_{n=1}^{\infty} n^{\gamma}\left|\int_0^1 \varphi_n(x)dx\right|^p<+\infty,$$

*when* $1+\gamma-\frac{3}{2}p<0.$

*If for any* $t\in[0,1]$ *and arbitrary* $(a_n)\in l_2$

$$H_n(p,\gamma,a,t)=O(1)B_n(p,\gamma,a)$$

*then, for every* $f\in Lip1$

$$\sum_{n=1}^{\infty} n^{\gamma}|C_n(f)|^p<+\infty,\quad 1+\gamma-\frac{3}{2}p<0.$$

**Theorem 2.** *Let* $(\varphi_n)$ *be an ONS on* $[0,1]$*. We choose the numbers* $p$ *and* $\gamma$ *so that* $1+\gamma-\frac{3}{2}p<0$*. Assume that for some* $(b_n)\in l_2$ *or* $t_0\in[0,1]$

$$\limsup_{n\to\infty}\frac{H_n(p,\gamma,b,t_0)}{B_n(p,\gamma,b)}=+\infty.$$

*Then there exists an* $f_0 \in Lip1$*, such that*

$$\sum_{k=1}^{\infty} k^{\gamma} |C_k(f_0)|^p = +\infty.$$

**Theorem 3.** *Any ONS* $(\varphi_n)$ *contains a subsystem* $\omega_k = \varphi_{n_k}$*, for which, for any* $f \in Lip1$*, when* $1 + \gamma - \frac{3}{2}p < 0$*, the following condition is fulfilled*

$$\sum_{k=1}^{\infty} k^{\gamma} |C_k(f)|^p < +\infty,$$

*where*

$$C_k(f) = \int_0^1 f(x)\omega_k(x)dx.$$

**REFERENCES**


1. Alexits, G. Convergence problems of orthogonal series. *Translated from the German by I. F¨older. International Series of Monographs in Pure and Applied Mathematics, Vol. 20, Pergamon Press, New York-Oxford-Paris* 1961, 350 pp.
2. Tsagareishvili, V. On special Fourier series. *Anal.Math.*, **43**, 4 (2017), 631–640.
3. Golubov, B.I. On Fourier series of continuous functions with respect to a Haar system (Russisn). *Izv. Akad. Nauk SSSR Ser. Mat.*, **28** (1964), 1271–1296.
4. Golubov, B.I. Absolute convergence of double series of Fourier-Haar coefficients for functions of bounded $p$-variation (Russian). *Izv. Vyssh. Uchebn. Zaved. Matematika*, **6** (2012), 3-13. translation in Russian Math. (Iz. VUZ), **56**, 6 (2012), 1-10.
5. Gogoladze, L., Meskhia, R. On the absolute convergence of trigonometric Fourier series. *Proc. A. Razmadze Math. Inst.*, **141** (2006), 29-40.
6. Volosivets, S.S., Golubov, B.I. Generalized absolute convergence of series of Fourier coefficients with respect to Haar type systems. *Russian Mathematics*, **62**, 1 (2018), 7–16.
7. Gogoladze, L., Tsagareishvili, V. Fourier coefficients of continuous functions. *Mathematical Notes*, **91**, 5-6 (2012), 645–656.


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Author(s) address(es):

Leri Gogoladze, Giorgi Cagareishvili
Department of Mathematics, Faculty of Exact and Natural Sciences
I. Javakhishvili Tbilisi State University
University str. 13, 0186 Tbilisi, Georgia
E-mail: lgogoladze1@hotmail.com, giorgicagareishvili7@gmail.com