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# THE MÖBIUS PHENOMENON IN GENERALIZED MÖBIUS-LISTING BODIES WITH CROSS SECTIONS OF ODD AND EVEN POLYGONS

Johan Gielis  Ilia Tavkhelidze

**Abstract**. In the study of cutting Generalized Möbius-Listing $GML_m^n$ bodies with polygons as cross section, it is well known that the Möbius phenomenon, whereby the cutting process yields only one body, occurs only in even polygons with an even number of vertices and sides, and only in the specific case when the knife cuts through the center of the polygon. This knife in this case cuts from vertex to vertex, vertex to side or side to side, cutting exactly two points on the boundary of the polygon. This knife is called a chordal knife, in connection to the chord cutting a circle. If the knife is a radial knife, i.e. it cuts only one point of the boundary, the Möbius phenomenon can occur both in odd and even polygons, but only when the radial knife cuts the center of the polygon.

**Keywords and phrases**: Generalized Möbius-Listing's bodies and surfaces, Möbius phenomenon, regular polygons, Gielis Transformations.

**AMS subject classification (2010)**: 53A05, 51B10.

## 1 Introduction.

In this article we use the following notations:

- Generalized Möbius-Listing $GML_m^n$ surfaces and bodies are toroidal structures obtained from cylinders whose cross sections have rotational symmetry $C_m$, e.g. regular polygons, and with the centers of all cross sections forming the basic line of the cylinder (Figure 1) [1]. Generalized Möbius-Listing Bodies $GML_m^n$ are defined by (1) or (2):

$$\begin{aligned} X(\tau,\psi,\theta,t) &= [R(\theta)+p(\tau,\psi)\cos(\tfrac{n\theta}{m})-q(\tau,\psi)\sin(\tfrac{n\theta}{m})]\cos(\theta),\\ Y(\tau,\psi,\theta,t) &= [R(\theta)+p(\tau,\psi)\cos(\tfrac{n\theta}{m})-q(\tau,\psi)\sin(\tfrac{n\theta}{m})]\sin(\theta),\\ Z(\tau,\psi,\theta,t) &= K(\theta)+p(\tau,\psi)\sin(\tfrac{n\theta}{m})+q(\tau,\psi)\cos(\tfrac{n\theta}{m}),\end{aligned} \tag{1}$$

or, alternatively

$$\begin{aligned} X(\tau,\psi,\theta,t) &= [R(\theta)+p(\tau,\psi)\cos(\psi+\tfrac{n\theta}{m})]\cos(\theta),\\ Y(\tau,\psi,\theta,t) &= [R(\theta)+p(\tau,\psi)\cos(\psi+\tfrac{n\theta}{m})]\sin(\theta),\\ Z(\tau,\psi,\theta,t) &= K(\theta)+p(\tau,\psi)\sin(\psi+\tfrac{n\theta}{m}).\end{aligned} \tag{2}$$

$X, Y, Z$ is the ordinary notation for space coordinates and $\tau, \psi, \theta$ are local coordinates where $\tau \in [-\tau^*, \tau^*]$, with $0 < \tau$ ; $\psi \in [0; 2\pi]$ and $\theta \in [0; 2\pi h]$ with $h \in \mathbb{R}$. The functions $K(\theta)$, $R(\theta)$, $p(\tau, \psi)$, $q(\tau, \psi)$, as well as parameter $\frac{n}{m}$ (defining twisting around the basic line), define simple movements.With these analytic representations also complex movements can be studied and decomposed into simple movements, a study initiated by Gaspar Monge. In case the cylinder is a strip and is given a half twist ($180^0$) before joining, the classic Möbius band results, a $GML_2^1$ surface. The half twist for $m = 2$ is $n = 1$. It is well known that cutting Möbius bands yields surprising results, and the study of cutting of $GML_m^n$ bodies showed a close link to the study of knots and links.

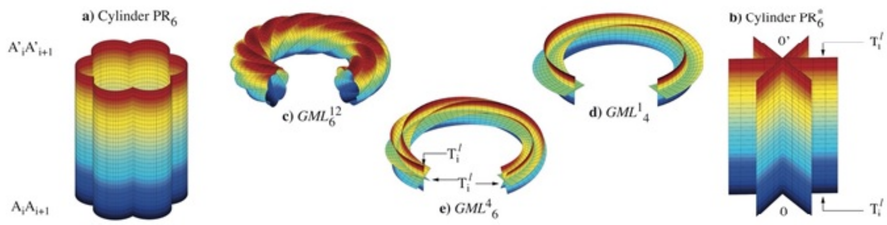


Figure 1: Identification of vertices $A'_iA'_{i+1}$ with $A_iA_{i+1}$ leading to prism, with twists leading to torus $GML_m^n$ with identification of vertices $T'_i$ with $T_i$

In the study of cutting $GML_m^n$ surfaces and bodies, cutting is performed with 1) a straight $d_m$ knife, which 2) cuts perpendicular to the polygonal cross section of the $GML_m^n$ surfaces and bodies, and 3) the knife cuts the $m$-polygon boundary exactly in two points or two times (depending on the thickness of the knife). For 3) there are three possibilities: the cut of the polygon can be from a vertex to a vertex $VV$, from a vertex to a side or edge $VS$, or from side to side $SS$. The precise orientation of this knife (and the positions where it cuts the boundary) is maintained during the complete cutting process, until the knife returns to its starting position, and the cutting is completed. Depending on the number of twists, one or more independent bodies result, related to the divisors of m. In Figure 2 a,b different colors indicate different bodies after cutting [2,3]. For some values

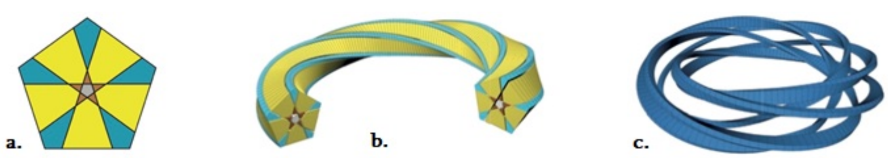


Figure 2: a - Cross section of (b);
b - A pentagonal $GML_5^n$ body with 4 different bodies after cutting;
c - One of the resulting structures for $GML_6^n$

of m and n a single body results after a full cutting of the $GML_m^n$ body, which displays the Möbius phenomenon of one-sided bodies. The necessary conditions for obtaining the Möbius phenomenon after cutting $GML_m^n$ surfaces or bodies are [1,4]:

1. The number has to be even ($m = 2, 4, 6, \ldots.$). The Möbius phenomenon never occurs when the polygon has an odd number of vertices and sides;

2. The knife has $m$ blades cutting all vertices with maximal length of the knife (i.e. from one vertex or side to the opposite one, then repeating this for every vertex or side);

3. The $d_m$ knife has to cut through the center (cuts are either vertex to opposite vertex, or side to side through center)

We show the necessary conditions under which the Möbius phenomenon occurs when m is an odd number ($m = 3; 5; 7, \ldots$).

**2 Radial and chordal knives.** The cutting process of the 3 dimensional $GML_m^n$ bodies could be related one-to-one to plane geometry whereby the knife is a drawing tool in plane geometry [2;3;5]. The $d_m$ knife is a line of infinite length (1), rotated m times or a divisor thereof. The analytic definition of $d$-knife (3) is a construction with straight lines, whereby the number of straight lines is either m or one of its divisors:

$$\sin\left(\alpha + \frac{2\pi i}{m}\right) x_i + \cos\left(\alpha + \frac{2\pi i}{m}\right) y_i + \delta = 0; i = 0, 1, \ldots, m-1; -\frac{\pi}{m} \leq \alpha \leq \frac{\pi}{m}, \quad (3)$$

with $\alpha$ the rotation parameter, and $\delta$ the translation parameter of this infinite line. For a hexagon (Figure 3), one can visualize the $m$ knives and their rotation for a $d_1$ knife cutting from vertex 1 to 4. The rotated knives on the right show the orientation of the knife for every case when it cuts $GML_m^n$ bodies or surfaces, with n=m twists. The point of the arrow coincides with the toroidal lines on the twisted $GML_m^n$ bodies or surfaces every $60^0$. Superposition of these six knives gives $d_{m=6}$ knife. If we now limit the knife in (3) to the half line, originating in the origin or center of the polygon, we obtain a radial knife (Figure 3). The name derives from radius versus the diameter in chordal knives. It cuts the polygon boundary only in one point. A radial knife $d_{rc}$ in Figure 3 is also a

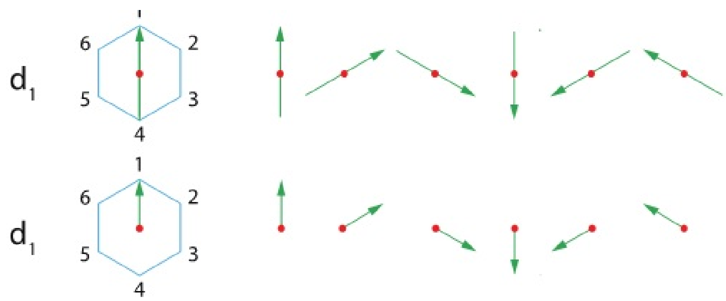


Figure 3: A chordal knife (top row) and radial knife (bottom)

half-line or ray, corresponding to the classical position vector, the most elementary and natural geometric object. Radial knives can either originate at the center ($d_{rc}$) or not $d_{r\bar{c}}$ and can be translated using $\delta$ and/or rotated using $\alpha$, the translation and rotation parameters in (1), respectively. The length of the $d_{rc}$ is defined as the length from origin to the perimeter but can be extended or made shorter, as long as the radial knife cuts only one point on the boundary.

Now, to show that the radial knife also results in the Möbius phenomenon when $m$ is an odd number, the strategy is to look only at the simplest possibility, which for $m$ even is a cut through the center or origin from a vertex to the opposite vertex, for all vertices. In this case the planar geometrical configuration is $m/2$. In 3D $GML_m^n$ surfaces and bodies the Möbius phenomenon occurs when $m$ is even and $n/2$. This is similar to the classic Möbius ribbon, which is a ribbon (a special case of a bilaterally symmetric shape with $m = 2$), twisted $180^0$ before closing.

Because of the equivalence of the cutting of $GML_m^n$ surfaces and bodies and their representations in plane geometry as in Figure 2, the Möbius condition is achieved when in planar view, all diagonals are used, in other words, when $d_m = m$. It is then required to have the same situation in odd polygons.

It is easy to see that this is very well possible with radial knives. Generalizing radial knives $d_{rc}$ for any symmetry, i.e. rotating the radial $d_{r1}$ knife to $m$ positions, results in $m$ radial knives for $m$-regular polygons, irrespective of whether $m$ is even or odd. But now $m$ radial knives are needed, in contrast to an amount of $m/2$ of chordal knives. In 3D $GML_m^n$ surfaces and bodies being cut with radial knives, this means that $n = m$ twists are needed to connect both ends of the cylinder. The result then is that the full cutting results in a single one-sided body, displaying the Möbius phenomenon, so we have:

**Proposition.** *If the knife cutting a $GML_n^n$ body is a radial knife with origin in the center of the polygonal cross sections and cuts all sides of the polygon with equal spacing the Möbius phenomenon will occur in both in odd and even polygons.*

**Corollary.** This holds true for any convex cross section with symmetry $C_m$ and with $m$ equally spaced points with $m$ a positive integer.

## REFERENCES

1. Tavkhelidze, I., Ricci, P.E. Some Properties of "Bulky" Links, Generated by Generalized Möbius-Listing's Bodies $GML_m^n$. *In: Modeling in Mathematics, Atlantis Press, Paris,* (2017), 159–185.

2. Gielis, J., Tavkhelidze, I. The general case of cutting of GML surfaces and bodies. 4 Open **3**, 7 (2020), 1-48, https://doi.org/10.1051/fopen/2020007.

3. Pinelas, S., Tavkhelidze, I. Analytic Representation of Generalized Möbius-Listing's Bodies and Classification of Links Appearing After Their Cut.*In International Conference on Differential avd Difference Equations and Applications, Springer* (2017), 477-493.

4. Tavkhelidze, I., Gielis, J. The process of cutting $GML_n^m$ bodies with $d_m$-knives. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.*, **32**, (2018), 67-70.

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Author(s) address(es):

Johan Gielis
University of Antwerp
Department of Bio-Engineering Sciences, Belgium
E-mail: johan.gielis@uantwerpen.be

Ilia Tavkhelidze
Department of Mathematics, Faculty of Exact and Natural Sciences
I. Javakhishvili Tbilisi State University
University str. 13, 0186 Tbilisi, Georgia
E-mail: ilia.tavkhelidze@tsu.ge