Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 34, 2020

ON 2-GENERATED FREE S_1^ω-ALGEBRAS

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Abstract. One and two-generated free MV-algebras are algebraically described in the variety, generated by perfect MV-algebras. Moreover, the MV-algebras are visualized by their ordered spectral spaces.

Keywords and phrases: MV-algebra, Perfect MV-algebra, Łukasiewicz logic.

AMS subject classification (2010): 06D35, 03B50, 03G20.

1 Introduction. MV-algebras are the algebraic counterpart of the infinite valued Łukasiewicz sentential calculus, as Boolean algebras are with respect to the classical propositional logic. In contrast with what happens for Boolean algebras, there are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of an MV-algebra) is different from $\{0\}$. Non-zero elements from the radical of A are called infinitesimals.

Subvarieties of MV-algebras have been studied in [9], [10], [11], [8], [14]. It is known that any such subvariety is generated by finitely many algebras, and explicit axiomatizations have been obtained. Notice that the free algebras over the subvarieties of MV-algebras have been described functionally in [14] using McNaughton functions [12].

The class of perfect MV-algebras does not form a variety and contains non-simple subdirectly irreducible MV-algebras [7]. It is worth stressing that the variety $\mathcal{V}(S_1^\omega)$, denoted by $\mathbf{MV(C)}$ in [6], generated by all perfect MV-algebras is also generated by a single MV-chain $C(\cong S_1^\omega)$ defined by Chang in [3]. We name by S_1^ω-*algebras* all the algebras from the variety generated by $S_1^\omega(\cong C)$ [6]. Notice that the Lindenbaum algebra of the logic L_P is an S_1^ω-algebra where L_P is the logic corresponding to the variety $\mathcal{V}(S_1^\omega)$. The perfect algebra $C(\cong S_1^\omega)$ has relevant properties. Indeed S_1^ω generates the smallest variety of MV-algebras containing non-boolean non-semisimple algebras. It is also subalgebra of any non-boolean perfect MV-algebra. The variety $\mathcal{V}(S_1^\omega)$ is selected from the variety $\mathbf{MV}$ of all MV-algebras by the identity $2(x^2) = (2x)^2$ [7].

The importance of the class of S_1^ω-algebras and the logic L_P can be perceived by looking further at the role that infinitesimals play in MV-algebras and Łukasiewicz logic. Indeed the pure first order Łukasiewicz predicate logic is not complete with respect to the canonical set of truth values $[0, 1]$, see [15], [1]. The Lindenbaum algebra of the first order Łukasiewicz logic is not semisimple and the valid but unprovable formulas are precisely the formulas whose negations determine the radical of the Lindenbaum algebra, that is the co-infinitesimals of such algebra. Hence, the valid but unprovable formulas generate the perfect skeleton of the Lindenbaum algebra. So, perfect MV-algebras, the variety generated by them and their logic are intimately related with a crucial phenomenon of the first order Łukasiewicz logic [2].

In this paper we give an algebraic description of one and two-generated free MV-algebras in the variety $\mathcal{V}(S_1^\omega)$ generated by the algebra S_1^ω that was introduced by Komori [11]. Moreover, we give ordered spectral spaces of the free algebras. In this paper we represent free algebras by means of subdirect product of a finite family of chains according to Panti's result in [14].

2 Preliminaries. We assume familiarity with MV-algebras; we refer to [3], [4], [13], [5] for all unexplained notions and claims.

An MV-algebra $A = (A, 0, \neg, \oplus)$ is an abelian monoid $(A, 0, \oplus)$ equipped with a unary operation $\neg$ such that $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, and $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus y)$. We set $1 = \neg 0$ and $x \odot y = \neg(\neg x \oplus \neg y)$ [3]. We shall write ab for $a \odot b$ and a^n for $\underbrace{a \odot \cdots \odot a}_{n \text{ times}}$, for given $a, b \in A$. Every MV-algebra has an underlying ordered structure, defined by $x \leq y$ iff $\neg x \oplus y = 1$. $(A; \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra: $xy \leq x \wedge y \leq x \vee y \leq x \oplus y$.

The unit interval of real numbers $[0, 1]$, endowed with the following operations: $x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), \neg x = 1 - x$, becomes an MV-algebra. It is well known that the MV-algebra $S = ([0, 1], \oplus, \odot, \neg, 0, 1)$ generates the variety $\mathbf{MV}$ of all MV-algebras, i. e. $\mathcal{V}(S) = \mathbf{MV}$.

The algebra S_1^ω-algebra (or C in Chang's notation), with generator $(0, 1)$, is isomorphic to $\Gamma(Z \times_{lex} Z, (1, 0))(= S_1^\omega)$, where Γ is Mundici's Γ Functor. Let $\mathcal{V}(S_1^\omega)$ be the variety, generated by perfect algebras. The intersection of all maximal ideals of an MV-algebra A, the radical of A, will be denoted by $Rad(A)$. Notice, that $\mathbf{MV(C)} = \mathcal{V}(C) = \mathcal{V}(S_1^\omega)$ where $S_1^\omega \cong C$ and $\mathcal{V}(S_1^\omega)$ is the variety, generated by S_1^ω-algebras.

Let us introduce some notations.

$S_1^{\omega(1)} = \Gamma(Z \times_{lex} Z, (1, 0)) = C$, $S_1^{\omega(2)} = \Gamma(Z \times_{lex} Z \times_{lex} Z, (1, 0, 0))$ where $(1, 0, 0) \in Z^3$ and $Z \times_{lex} Z$ is the lexicographic product of Z.

The class of MV-algebras forms a category where the objects of this category are MV-algebras and morphisms between MV-algebras are homomorphisms. A topological space X is said to be an *MV-space* iff there exists an MV-algebra A such that $Spec(A)$ (= the set of prime filters of the MV-algebra A equipped with spectral topology) and X are homeomorphic. It is well known that $Spec(A)$ with the specialization order R on X (which coincides with the inclusion between prime filters) forms a root system.

3 1-generated free S_1^ω-algebra. In this section we give the descriptions of 1-generated free S_1^ω-algebras in the variety $\mathcal{V}(S_1^\omega)$. It is easy to prove the following

Theorem 1. *1) $S_1^{\omega(2)}$ is generated by 2 generators: $(0, 0, 1), (0, 1, 0)$;*

2) $S_1^{\omega(1)}$ is a homomorphic image of $S_1^{\omega(2)}$.

Theorem 2. *[6]. 1-generated free S_1^ω-algebra $F_{\mathcal{V}(S_1^\omega)}(1)$ is isomorphic to $(S_1^\omega)^2$ with free generator $g = ((0,1),(1,-1))$.*

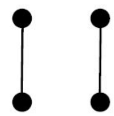

Figure 1: The ordered MV-space of the S_1^ω-algebra $(S_1^\omega)^2$

4 2-generated free S_1^ω-algebra. In this section we give the descriptions of 2-generated free S_1^ω-algebras in the variety $\mathcal{V}(S_1^\omega)$.

For the sake of simplicity let us introduce the following notations for the generating elements of the algebra $S_1^{\omega(2)}$: $c_1 = (0,0,1), c_2 = (0,1,0)$. Notice, that S_1^ω-algebra $S_1^{\omega(2)}$ is generated by two generators c_1 and c_2.

The algebra A is perfect if $A = Rad^*(A) = Rad(A) \cup \neg Rad(A)$, where $\neg Rad(A) = \{\neg x : x \in Rad(A)\}$ is the intersection of all maximal filters of A.

Theorem 3. *2-generated free S_1^ω-algebra $F_{\mathcal{V}(S_1^\omega)}(2)$ is isomorphic to $(Rad^*((S_1^{\omega(2)})^2))^{2^2}$ with free generators $g_1 = ((c_1,c_2), \neg(c_1,c_2), (c_1,c_2), \neg(c_1,c_2))$ and $g_2 = ((c_2,c_1),(c_2,c_1), \neg(c_2,c_1), \neg(c_2,c_1))$.*

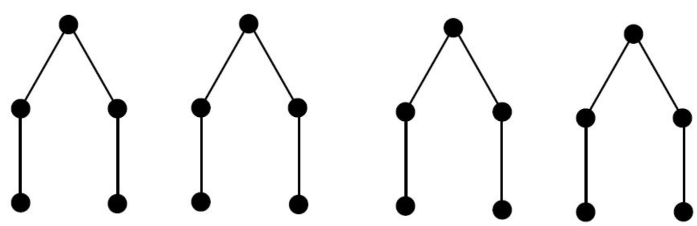

Figure 2: The ordered MV-space of the S_1^ω-algebra $(Rad^*((S_1^{\omega(2)}))^2)^{2^2}$

5 Conclusions. In this paper, one and two-generated free MV-algebras are algebraically described in the variety generated by perfect MV-algebras.

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Received 21.05.2020; revised 28.07.2020; accepted 22.09.2020.

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