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STUDY OF THE BASIC BOUNDARY VALUE PROBLEMS OF STATICS OF THE THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES AND MICROROTATION BY THE POTENTIAL METHOD

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Abstract. The paper deals with the basic boundary value problems of statics for thermoelastic isotropic microstretch materials with microtemperatures and microrotation. For the homogeneous system of partial differential equations of statics, the fundamental matrix is constructed explicitly in terms of elementary functions. By means of the fundamental matrix is constructed the corresponding volume and layer potentials, and their mapping properties are investigated. The basic Dirichlet and Neumann type boundary value problems are reduced to the corresponding system of singular integral equations, and the existence theorems of solutions are proved.

Keywords and phrases: Microstretch materials, microtemperatures, potential method.

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1 Introduction. The mathematical model of a linear theory of thermodynamics for microstretch elastic solids with microtemperatures, using the results established by Grot [1] have been proposed by Ieşan [2]. This theory introduces three extra degrees of freedom over the theory presented in [3]. An interesting of microrotation vector with the microtemperatures even for isotropic bodies. This effect is different from the elastical theory of Cosserat thermoelasticity for isotropic bodies [4], where the microrotation vector is independent of the thermal field. The basic boundary value problems for the static equations of the theory of thermoelasticity for isotropic materials with microstructure and microtemperature are investigated in [5].

2 Basic equations and boundary value problems. The homogeneous system of static equation of the thermoelasticity theory of microstrech materials with microtemperatures and microdilatation in the case of isotropic bodies read as [1]

$$(\mu+\varkappa)\Delta u+(\lambda+\mu)\,grad\,div\,u-\varkappa\,rot\,\omega+\mu_0\,grad\,v-\beta_0\,grad\,\vartheta=0, \tag{1}$$

$$\varkappa_6\Delta w-\varkappa_2 w+(\varkappa_4+\varkappa_5)\,grad\,div\,w-\varkappa_3\,grad\,\vartheta=0, \tag{2}$$

$$\gamma\Delta\omega-2\varkappa\omega+\varkappa R(\partial)u-\mu_1 R(\partial)w=0, \tag{3}$$

$$a_0\Delta v-\eta v-\mu_0\,div\,u-\mu_2\,div\,w+\beta_1\vartheta=0, \tag{4}$$

$$\varkappa_7\Delta\vartheta+\varkappa_1\,div\,w=0, \tag{5}$$

where γ, λ, μ, $\varkappa$, η, β_0, β_1, μ_0, μ_1, μ_2, a_0, $\varkappa_j, j=1,2,...,7$ are the real constants characterizing the mechanical and thermal properties of the body, Δ is the Laplace operator,

$u = (u_1, u_2)$ is the displacement vector, $w = (w_1, w_2)$ is the microtemperature vector, ω is the microrotation function, v is the microdilatation function, ϑ is the temperature, measured from a fixed absolute temperature T_0 ($T_0 > 0$), the superscript $(\cdot)^\top$ denotes transposition operation,

$$R(\partial)u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad R(\partial)w = \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}.$$

Let $\Omega^+ \subset R^2$ be a bounded domain with boundary $\partial\Omega$. We denote $\Omega^- = R^2 \setminus \overline{\Omega}^+$.

Problem. Find in the domain Ω^+ (Ω^-) such a regular vector $U = (u, w, \omega, v, \vartheta)^\top \subset C^2(\Omega^\pm) \cap C^1(\overline{\Omega}^\pm)$ that satisfies in the domain the system of differential equations (1)-(5), and on the boundary $\partial\Omega$, satisfies one of the following boundary conditions

$(I)^\pm$ **(The Dirichlet problem)**

$$\{U(z)\}^\pm = f(z), \quad z \in \partial\Omega, \tag{6}$$

$(II)^\pm$ **(The Neumann problem)**

$$\{P(\partial, n)U(z)\}^\pm = f(z), \quad z \in \partial\Omega, \tag{7}$$

where $f = (f^{(1)}, f^{(2)}, f_3, f_4, f_5)^\top$, $f^{(j)} = (f_1^{(j)}, f_2^{j})^\top$, $j = 1, 2$, are given vector-functions and f_l, $l = 3, 4, 5$ are given functions on the boundary $\partial\Omega$, $n(z)$ is the outward normal unit vector passing at a point $z \in \partial\Omega$ with respect to the domain Ω^+, $P(\partial, n)U$ is the generalized thermo-stress vector.

In the case of the exterior problems for the domain Ω^- the vector $U = (u, w, \omega, v, \vartheta)^\top$ should satisfy the following decay conditions at infiniti

$$U(x) = O(|x|^{-1}), \quad \frac{\partial}{\partial x_k} U(x) = o(|x|^{-1}), k = 1, 2.$$

3 Existence results for boundary value problems. Let us denote by $L(\partial)$ the matrix differential operator of order 7×7, generated by the left hand side expressions in system (1)-(5). Assume that

$$L^*(\partial)U := L^\top(-\partial)U.$$

Let us introduce the generalized single and double layer potentials, and the Newton type volume potential

$$V(\varphi)(x) = \int_{\partial\Omega} \Gamma(x - y)\, \varphi(y)\, d_y S, \quad x \in \mathbb{R}^2 \setminus \partial\Omega, \tag{8}$$

$$W(\varphi)(x) = \int_{\partial\Omega} \left[\mathcal{P}^*(\partial_y, n(y))\Gamma^\top(x - y)\right]^\top \varphi(y)\, d_y S, \quad x \in \mathbb{R}^2 \setminus \partial\Omega, \tag{9}$$

$$N_{\Omega^\pm}(\psi)(x) = \int_{\Omega^\pm} \Gamma(x - y)\, \psi(y)\, dy, \quad x \in \mathbb{R}^2, \tag{10}$$

where $\Gamma(\cdot)$ is the fundamental matrix, $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_7)^\top$ is a density vector-function defined on $\partial\Omega$, while a density vector-function $\psi = (\psi_1, \psi_2, \cdots, \psi_7)^\top$ is defined on $\Omega^\pm$, $P^*(\partial_y, n(y))$ is the boundary differential operator, corresponding to $L^*(\partial)$.

Theorem 1. *For any $g \in C^{0,\delta'}(\partial\Omega)$, $h \in C^{1,\delta'}(\partial\Omega)$, and any $x \in \partial\Omega$*

$$
\begin{aligned}
&\{ V(g)(x) \}^\pm = V(g)(x) = \mathcal{H}\, g(x),\\
&\{ P(\partial_x, n(x))\, V(g)(x) \}^\pm = \big[\mp 2^{-1} I_7 + \mathcal{K} \big]\, g(x),\\
&\{ W(g)(x) \}^\pm = \big[\pm 2^{-1} I_7 + \mathcal{N} \big]\, g(x),\\
&\{P(\partial_x, n(x))\, W(h)(x) \}^+ = \{ P(\partial_x, n(x))\, W(h)(x) \}^- = \mathcal{L}\, h(x),
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{H}\, g(x) := \int_{\partial\Omega} \Gamma(x-y)\, g(y)\, d_yS\,,\\
&\mathcal{K}\, g(x) := \int_{\partial\Omega} \big[P(\partial_x, n(x))\, \Gamma(x-y) \big]\, g(y)\, d_yS\,,\\
&\mathcal{N}\, g(x) := \int_{\partial\Omega} \big[P^*(\partial_y, n(y))\, \Gamma^\top(x-y) \big]^\top g(y)\, d_yS\,,\\
&\mathcal{L}\, h(x) := \lim_{\Omega^\pm \ni z \to x \in \partial\Omega} P(\partial_z, n(x)) \int_{\partial\Omega} \big[P^*(\partial_y, n(y))\, \Gamma^\top(z-y) \big]^\top h(y)\, d_yS\,.
\end{aligned}
$$

Theorem 2. *Let $\partial\Omega \in C^{2,\nu}$ and $f \in C^{1,\tau}(\partial\Omega)$ with $0 < \tau < \nu \le 1$. Then the boundary value problem $(I)^+$ is uniquely solvable in the space $C^{1,\tau}(\overline{\Omega^+})$ and the solution is represented by the double layer potential $W(h)$ defined by (9), where density $h \in C^{1,\tau}(\partial\Omega)$ is a unique solution of the integral equation $\big[2^{-1} I_7 + \mathcal{N} \big]\, h = f$.*

Theorem 3. *Let $\partial\Omega \in C^{2,\nu}$ and $f \in C^{1,\tau}(\partial\Omega)$ with $0 < \tau < \nu \le 1$. Then the boundary value problem $(I)^-$ is uniquely solvable in the space $C^{1,\tau}(\overline{\Omega}^-)$ and the solution is represented by linear combination of the double and single layer potentials.*

Theorem 4. *Let $\partial\Omega \in C^{1,\nu}$ and $f \in C^{0,\tau}(\partial\Omega)$ with $0 < \tau < \nu \le 1$. The null space of the singular integral operator $-2^{-1} I_7 + \mathcal{K} : C^{0,\tau}(\partial\Omega) \to C^{0,\tau}(\partial\Omega)$ corresponding to the homogeneous interior Neumann boundary value problem $(II)_0^+$ have the dimension equal to 5. Moreover, the vectors*

$$
\begin{aligned}
\varphi^{(1)}(x) =& (-x_2,\ x_1,\ 0,\ 0,\ 0,\ 0,\ 0)^\top,\\
\varphi^{(2)}(x) =& (1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)^\top,\\
\varphi^{(3)}(x) =& (0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0)^\top,\\
\varphi^{(4)}(x) =& (0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0)^\top,\\
\varphi^{(5)}(x) =& (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1)^\top,
\end{aligned}
$$

restricted into the surface $\partial\Omega$ *represent a basis* $\varphi^{(k)}(x)$, $k = 1, 2, \cdot, 5$ *of the null space of the adjoint singular integral operator* $-2^{-1}I_7 + \mathcal{N}^*$.

Theorem 5. *Let* $\partial\Omega \in C^{1,\nu}$ *and* $f \in C^{0,\tau}(\partial\Omega)$ *with* $0 < \tau < \nu \leq 1$. *Then the boundary value problem* $(II)^-$ *is uniquely solvable in the space* $C^{1,\tau}(\overline{\Omega^-})$ *and the solution is represented by the single layer potential.*

REFERENCES

1. Grot, R. Thermodynamics of a continuum mechanics. *Internat. J. Engrg. Sci.*, **3** (1969), 801–814.
2. Ieşan, D. Thermoelasticity of bodies with microstructure and microtemperatures. *Int. J. Solids Struct.*, **44** (2007), 8648–8662.
3. Ieşan, D. On a theory of micromorfic elastic solids with microtemperatures. *J. Thermal Stresses.*, **24** (2001), 737–752.
4. Cosserat, E., Cosserat, F. Therie Des Corps Deformablees. *Herman, Paris.*, 1909.
5. Giorgashvili, L., Zazashvili, S. Mathematical problems of thermoelasticity of bodies with microstructures and microtemperatures, *Transactions of A. Razmadze Mathematical Institute.*, **171**, 3 (2017), 350–378.

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