Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 34, 2020

FREE CONSTRUCTIONS IN CATEGORIES OF EXPONENTIAL *MR*-GROUPS

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Abstract. In the present paper, for an arbitrary associative ring R with unity we define a category of R-groups in three different ways. The key idea consists in realizing a tensor completion of an MR-group in the form of a concrete structure using free products with union. As a result, the description of free MR-groups and free MR-products is obtained in terms of free group structures.

Keywords and phrases: Lyndon R-group, MR-group, Hall R-group, tensor completion, free MR-group, free product of MR-groups.

AMS subject classification (2010): 20B07.

The concept of an exponential R-group, where R is an arbitrary associative ring with identity, was introduced by Lyndon in [1]. In [2] Myasnikov and Remeslennikov refined the concept of an R-group by adding an extra axiom. Specifically, the new concept of an exponential R-group is direct generalization of the concept of an R-module to noncommutative groups. In the honor of Myasnikov, R-groups with an extra axiom were called in [3] MR-groups (R stands for a ring). The role of the tensor extension of a ring of scalars for modules is well known. An exact analogue of this construction for an arbitrary MR-group – tensor completion – was defined in [2]. A method for constructing the tensor completion of a given MR-group was proposed in [4]. A systematic study of MR-groups was begun in [3-10]. Note that the results of these works turned out to be useful in solving well-known problems of Tarski.

Following [2], we recall the basic definitions and facts. Fix an arbitrary associative ring R with identity 1, and let G be a fixed group. The result of the action of $\alpha \in R$ on $g \in G$ is written as g^α. Consider the following axioms:

$$g^1 = g, \quad g^0 = e, \quad e^\alpha = e, \tag{1}$$

$$g^{\alpha+\beta} = g^\alpha g^\beta, \quad g^{\alpha\beta} = (g^\alpha)^\beta, \tag{2}$$

$$(h^{-1}gh)^\alpha = h^{-1}g^\alpha h. \tag{3}$$

$$(MR\text{-axiom}) \qquad \forall\, g,h \in G, \quad \alpha \in R, \quad [g,h] = e \longrightarrow (gh)^\alpha = g^\alpha h^\alpha, \tag{4}$$

where $[g,h] = g^{-1}h^{-1}gh$.

Definition 1 (see [2])**.** A group G is called an **exponential Lyndon R-group** if axioms (1)–(3) hold and is called an **exponential MR-group** if axioms (1)–(4) hold.

Let $\mathcal{L}_R$ and $\mathcal{M}_R$ denote the classes of all exponential Lyndon R-group and all MR-group respectively. Obviously, $\mathcal{L}_R \supseteq \mathcal{M}_R$. There are examples showing that this inclusion in strict (see [2, 11]). Moreover, every abelian MR-group is a R-module, and vice versa. The majority of natural examples of R-groups are in the class $\mathcal{M}_R$. For example, a free Lyndon R-groups is an MR-group, a unipotent group over a field K of zero characteristic is an MK-group an arbitrary pro-p-group is an $M\mathbb{Z}_p$-group over the ring $\mathbb{Z}_p$ of p-adic integers, and so on (for other examples, see [2]).

The concepts of an MR-subgroup MR-generatedness, normal MR-subgroup, etc., are introduced in the standard manner.

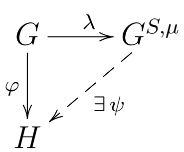
Clearly, this class is a quasi-variety in the signature $\langle \cdot,\ ^{-1}, f_\alpha \mid \ \alpha \in R\rangle$, where f_α is the unary operation of raising to the power α, i.e. $f_\alpha(g) = g^\alpha$, and a free MR-group, and R-homomorphism, etc., are defined in this class (see [10]). Specifically, a homomorphism of R-groups $\varphi : G \to G^*$ is called a R-**homomorphism** if $\varphi(g^\alpha) = \varphi(g)^\alpha$ for any $g \in G$ and $\alpha \in R$.

It was shown in [2] that a key role in the study of exponential MR-groups is played by the operation of tensor completion. It naturally generalizes the extension of a ring of scalars for modules to the noncommutative case.

Definition 2 (see [2])**.** Let G be an MR-group, let $\mu : R \to S$ be a ring homomorphism. Then an MS-group $G^{S,\mu}$ is called the **tensor** S-**completion** of the MR-group G if $G^{S,\mu}$ satisfies the following universal properties:

(a) there exists an R-homomorphism $\lambda : G \to G^{s,\mu}$ such that $\lambda(G)$ S-generates $G^{s,\mu}$, i.e. $\langle \lambda(G)\rangle_S = G^{s,\mu}$;

(b) for any MS-group H and any R-homomorphism $\varphi : G \to H$ compatible with μ (i.e. $\varphi(g^\alpha) = \varphi(g)^{\mu(\alpha)}$) there exists an S-homomorphism $\psi : G^{s,\mu} \to H$ such that the following diagram commutes:



$(\varphi = \lambda\psi)$.

Note that, if G is an abelian MR-group, then $G^{S,\mu}$ is abelian as well, i.e. an S-module and $G^{S,\mu} \cong G \underset{R}{\otimes} S$ is the tensor product of the R-module G by the ring S. It was proved in [2] that, for any group $G \in \mathcal{M}_R$ and any homomorphism $\mu : R \to S$, there always exits a tensor completion of $G^{S,\mu}$ and it is unique up to an R-isomorphism. A method, based on combinatorial group theory was proposed for constructing a tensor completion in [8]. In what follows, the ring homomorphism $\mu : R \to S$ is fixed, so, instead of $G^{S,\mu}$, we write G^S. In applications, μ is most frequently an embedding of rings, but, even in this case, the R-homomorphism $\lambda : G \to G^S$ is not always an embedding (see [9]). Obviously, the class $\mathcal{M}_R$ ($\mathcal{L}_R$) of all exponential MR-groups (Lyndon R-groups) is a category in which morphisms are R-homomorphisms of groups. In the language of category theory, the above-described completion operation is a tensor completion functor.

In the category $\mathcal{M}_R$ and $\mathcal{L}_R$,it is convenient to perform many constructions step by step, gradually defining powers. This leads to the concept of a partial R-group. A group G is called a **partial** MR**-group** if the operation of raising to power is defined for some pairs (g, α), but not necessarily for all pairs; moreover, if one part of an equality in axioms (1)–(4) is defined, then the other part is defined as well and they satisfy axioms (1)–(4) in the definition of an exponential MR-group. The class of partial MR-groups is denoted by $\mathcal{P}_R$. For example, if R is a subring of a ring S, then any R-group is a partial S-group.

In the category $\mathcal{P}_R$, we consider a class of groups $\mathcal{P}_R^0$. By definition, a group G from $\mathcal{P}_R$is said to belong to $\mathcal{P}_R^0$ if it satisfies the following conditions:

1) for any maximal abelian subgroup M of G and for any $x \notin M$, it is true that $M \cap M^x = e$;

2) the canonical homomorphism $j : M \to M \underset{R}{\otimes} R$ is an embedding.

For example, for any ring $R \supseteq \mathbb{Z}$, free groups are partial MR-groups. Clearly, these groups belong to the class $\mathcal{P}_R^0$.

Theorem 1 (is announced in [9])**.** *Suppose that $\mathbb{Z}$ is a subgroup of ring R, G is a group of class – $\mathcal{P}_R^0$ and there are no elements of order 2 in G and R^+ (the additive group of the ring). Then G is true, i.e., the canonical mapping $\lambda : G \to G^R$is an embedding.*

Nilpotent R-groups. Let $c > 1$ be a natural number. Denote by $\mathcal{N}_{c,R}$ the category of nilpotent R-groups of nilpotence c the class $\mathcal{L}_R$, i.e. of the R-groups where the identity $\forall\, x_1, \dots, x_{c+1}[x_1, \dots, x_{c+1}] = e$ is fulfilled, and by $\mathcal{N}_{c,R}^0$ the category of nilpotent MR-groups of step c. The structure of R-groups without the axiom of choice (MR) is very complicated and that's why only the MR-group is studied in most of the works. In the rest of this paper only the MR-groups will be considered.

Exponential nilpotent R-groups over the binomial ring R, introduced by P. Hall [12], are MR-groups. Let us denote the variety of nilpotent Hall R-group of class $\leq c$ by $\mathcal{HN}_{c,R}$.

In [3] it if shown that the structure of groups from $\mathcal{N}_{c,R}$ is very different from the structure of Hall R-groups from $\mathcal{HN}_{c,R}$.

Free constructions. In the category $\mathcal{M}_R$, the concept of a free MR-group $F_R(X)$ with a free MR-generating set X and the concept of free MR-product $\underset{R}{*}G_i$ of groups G_i, $i \in I$, are introduced in a standard manner. In these definitions, the homomorphisms are to be replaced by R-homomorphisms.

Theorem 2. *For any X and R, where R contains the ring of integers $\mathbb{Z}$, a free MR-group $F_R(X)$ exists and is unique up to an R-isomorphism, and $F_R(X) = F(X)^R$.*

Theorem 3. *Let R be a ring, containing the ring of integers $\mathbb{Z}$, and let G_i be a set of MR-groups, $i \in I$. Then*

1) $\underset{R}{*} G_i = (*G_i)^R$;

2) *the canonical mapping* $\lambda : G_i \to (*G_i)^R$ *is an embedding.*

Theorem 4. *The class* $\mathcal{P}_R^0$ *is closed under free MR-products.*

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Received 29.05.2020; accepted 11.09.2020.

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