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# BAYSIAN CONSISTENT CRITERIA FOR HYPOTHESES TESTING

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**Abstract**. Existence of Bayesian consistent criteria for hypotheses testing is proved.

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**1 Introduction.** Let $X$ be a separable Hilbert space, let $B(X)$ be a Borel $\sigma$-algebra on it, and let $\{\mu_a,\ a \in M\}$ be a family of Gaussian measures on $B(X)$ with different mean values $a \in M$ and an identical correlation operator $B$. The set of hypotheses is the set of means $a \in M$. The set $M$ is a convex compact in $X$, and a priori measure is positive on every open set.

**2 Content.** Suppose that on $M$ it is possible to introduce the norm

$$||a||_V = \sqrt{(V^{-1}a, a)}, \tag{1}$$

where $V$ is some completely continuous symmetric positive operator. Let on a $\sigma$-algebra, generated by Borel sets of $M$ with respect to the norm $||\cdot||_V$, given a priori measure $\theta(da)$.

We will consider the Bayesian criteria for the unknown mean $a_0 \in M$ :

$$a_n^*(x) = \frac{\int a \exp\{(C_n a, x) - \frac{1}{2}(C_n a, a)\}\theta(da)}{\int \exp\{(C_n a, x) - \frac{1}{2}(C_n a, a)\}\theta(da)}, \tag{2}$$

where $x$ is an observation, and $C_n$ is some sequence of operators.

**Theorem 1.** *Let the following conditions be fulfilled:*

*1) $M$ is compact in metric (1);*

*2) $\theta(\{a : ||a - a_0||_V \le \epsilon\}) > 0$, $\epsilon > 0$ and besides, there is a non-negative operator $V_1$ such that*

$$\int \exp\{(V_1 a, a)\}\theta(da) < \infty$$

*and*

$$(C_n B C_n b, b) \le (V_1 b, b), \quad b \in M;$$

*3) there is a limit $\lim_{n\to\infty} (C_n a, x)$ under the measure $\mu_0$;*

*4) $\lim_{n\to\infty} \inf_{||a||_V > \epsilon} (C_n a, a) = +\infty$;*

*5) for all $\epsilon > 0$ there are $\alpha > 1$ and $\epsilon_1 < \epsilon$ such that*

$$\varliminf_{n\to\infty}\Big[\inf_{||\omega||_V>\epsilon}(C_n\omega,\omega)\sup_{||\omega||_V\le\epsilon_1}(C_n\omega,\omega)\Big] > \alpha.$$

*Then*

$$\mu_{a_0}\big(\{\omega : ||a_n^* - a_0||_V > \epsilon\}\big) \to 0 \ \ as \ \ n\to\infty.$$

*Proof.* We introduce the following notation $\omega = a - a_0$, $x = y + a_0$, $\omega_n^* = a_n^* - a_0$. It is clear that

$$\omega_n^*(x) = \frac{\int \omega\varrho_n(\omega)\theta(da)}{\int \varrho_n(\omega)\theta(da)}, \tag{3}$$

where

$$\varrho_n(\omega) = \exp\{(C_n\omega, x) - \frac{1}{2}(C_n\omega,\omega)\}.$$

Therefore,

$$||\omega_n^*||_V = \frac{\int_{||\omega||_V\le\epsilon}||\omega||_V\varrho_n(\omega)\theta(da)}{\int \varrho_n(\omega)\theta(da)} + \frac{\int_{||\omega||_V>\epsilon}||\omega||_V\varrho_n(\omega)\theta(da)}{\int \varrho_n(\omega)\theta(da)}. \tag{4}$$

Assuming $\widetilde{M} = \{\omega : a\in M\}$, $c = \sup\limits_{\omega\in\widetilde{M}}||\omega||_V$ and

$$H_n(x) = \frac{\int_{||\omega||_V>\epsilon}\varrho_n(\omega)\theta(da)}{\int \varrho_n(\omega)\theta(da)}.$$

From (4) we get

$$||\omega_n^*||_V \le \epsilon + cH_n(x). \tag{5}$$

To complete the proof, it suffices to prove that the multiplier $H_n(x)$ in (5) tends to zero with respect to the measure $\mu_{a_0}$ as $n\to\infty$. According to (5) we estimate the $H_n(x)$ as follows:

$$\begin{aligned} H_n(x) &\le \frac{\int_{||\omega||_V>\epsilon}\exp\{(C_n\omega,x) - \frac{1}{2}\inf\limits_{||\omega||_V>\epsilon}(C_n\omega,\omega)\}\theta(da)}{\int_{||\omega||_V\le\epsilon_1}\exp\{(C_n\omega,x) - \frac{1}{2}\sup\limits_{||\omega||_V\le\epsilon_1}(C_n\omega,\omega)\}\theta(da)} \\ &\le \exp\{-\frac{1}{2}\frac{\alpha-1}{\alpha}\inf_{||\omega||_V>\epsilon}(C_n\omega,\omega)\}g_n(x), \end{aligned} \tag{6}$$

where

$$g_n(x) = \frac{\int_{||\omega||_V>\epsilon}\exp\{(C_n\omega,x)\}\theta(da)}{\int_{||\omega||_V\le\epsilon_1}\exp\{(C_n\omega,x)\}\theta(da)}.$$

Further, using the Fatou theorem and condition 3), we have

$$\lim_{n\to\infty}\int_{||\omega||_V\le\epsilon_1}\exp\{(C_n\omega,x)\}\theta(da) \ge \int_{||\omega||_V\le\epsilon_1}\exp\{\lim_{n\to\infty}(C_n\omega,x)\}\theta(da).$$

Therefore, to complete the proof, it suffices to verify that the function

$$\int_{||\omega||_V>\epsilon} \exp\{(C_n\omega, x)\}\theta(da)$$

is bounded with respect to the measure $\mu_{a_0}$. For this we prove the boundedness of

$$\overline{\lim_{n\to\infty}}\, E_{\mu_{a_0}} \int \exp\{(C_n\omega, x)\}\theta(da).$$

Indeed, changing the order of integration and taking into account condition 2), we obtain

$$\overline{\lim_{n\to\infty}} \int\int \exp\{(C_n\omega, x)\}\mu_{a_0}(dx)\theta(da) < \infty.$$

Thus, $g_n(x)$ is bounded with respect to the measure $\mu_{a_0}$. Consequently, from relation (6), by virtue of condition 4), it follows that $H_n(x)$ converges to zero with respect to the measure $\mu_0$ as $n \to \infty$. $\square$

Let $e_1,\ e_2,\ \ldots, e_n,\ \ldots$ be eigenvectors and $\lambda_1,\ \lambda_2,\ \ldots, \lambda_n,\ \ldots$ be corresponding eigenvalues of the operator $B$. The following theorem is true.

**Theorem 2.** *Let the following conditions be fulfilled:*

*1) there exists a sequence of positive numbers $\gamma_n \to \infty$ as $n \to \infty$ such that for each $a, a_0 \in M$ and $\forall\epsilon > 0$*

$$a)\ \sum_{k=1}^{\infty} \frac{(a-a_0, e_k)^2}{\lambda_k\gamma_k} < \infty, \qquad \inf_{||a-a_0||_V>\epsilon} \psi_n(a-a_0) = +\infty,$$

*where*

$$\psi_n(x) = \frac{1}{\sqrt{\gamma_n}} \sum_{k=1}^{\infty} \frac{(x, e_k)^2}{\lambda_k};$$

*b) there are $\alpha > 1$ and $\delta < \epsilon$ such that for sufficiently large $n$*

$$\inf_{||a-a_0||_V>\epsilon} \psi_n(a-a_0) \geq \alpha \sup_{||a-a_0||_V\leq\delta} \psi_n(a-a_0);$$

*2) $M$ is compact in metric $||\cdot||_V$;*

*3) $\forall\epsilon > 0$ the equality $\nu(\{a : ||a-a_0||_V \leq \epsilon\}) > 0$ is true, where $\nu(\cdot)$ is a priori measure on the $\sigma$-algebra, generated by Borel sets in $M$.*

*Then one can indicate a sub-sequence $n_i$ for which*

$$a^*_{n_i}(x) = \frac{\int a e^{-\varphi_{n_i}(a-a_0)}\nu(da)}{\int e^{-\varphi_{n_i}(a-a_0)}\nu(da)}$$

*will be a consistent criterion for mean $a_0 \in M$.*

## REFERENCES

1. Ibramkhalilov, I., Skorokhod, A. Consistent Estimates of Parameters of Random Processes (Russian). *Naukova Dumka*, Kiev, 1980.

2. Zerakidze, Z., Mumladze, M. Statistical Structures and Consistent Criteries for Checking Hypotheses. *Deutschland-Lambert Academic Publishing*, Saarbrucken, 2015.

3. Zerakidze, Z., Purtukhia, O. The consistent criteria for hypotheses testing. *Ukrainian Mathematical Journal*, **71**, 4 (2019), 486-501.

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