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BAYESIAN CONSISTENT CRITERIA FOR THE WIENER PROCESS

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Abstract. Existence of Bayesian consistent criteria for the Wiener process is investigated.

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1 Introduction. We observe the process $x(t,\omega)=\varphi(t)+W(t,\omega)$, where $W(t)$ is the standard Wiener process and $\varphi\in M$ (M is some set from $C([0,\infty))$). The set of hypotheses is unknown mean values $\varphi(t)$ of the observed process. The purpose of this article is to indicate the conditions under which for $\varphi\in M$ there exists a consistent criteria for hypothesis testing.

2 Content. Let μ_ψ be a measure on $C([0,\infty))$, corresponding to the process $\varphi(t)+W(t)$, μ_φ^T is a measure on $C([0,T])$, corresponding to $x(t)$, $t\in[0,T]$. Suppose that for all $T>0$ the measure μ_φ^T is absolute continuous with respect to the measure μ_0^T, corresponding to $W(t)$, $t\in[0,T]$, that is, the following conditions are true:

1) for $\varphi\in M$, $\varphi(0)=0$, there exists $\varphi^{'}(t)$ and $\forall T>0$ if

$$\int_0^T[\varphi^{'}(t)]^2dt<\infty,\ then\ \mu_\varphi^T\ll\mu_0^T, \tag{1}$$

at that

$$\frac{\mu_\varphi^T}{\mu_0^T}[x(\cdot)]=\exp\left\{\int_0^T\varphi^{'}(t)dt-\frac{1}{2}\int_0^T[\varphi^{'}(t)]^2dt\right\}. \tag{2}$$

If (1) is satisfied, then for the orthogonality of the measures μ_{φ_1} and μ_{φ_1} it is necessary and sufficient that

$$\int_0^T[\varphi_1^{'}(t)-\varphi_2^{'}(t)]^2dt=0. \tag{3}$$

Next, suppose the following condition holds:

2) there is a differentiable function $g(t)$, such that $g(t)\to0$ as $t\to\infty$ and

$$\int_0^\infty g^2(t)[\varphi'(t)]^2dt<\infty \tag{4}$$

uniformly with respect to φ, and for $\varphi_1\neq\varphi_2$, $\varphi_1,\varphi_2\in M$:

$$\int_0^\infty g^2(t)[\varphi_1{}'(t)-\varphi_2{}'(t)]^2dt=+\infty.$$

Consider the quadratic functional

$$\Phi(t,\varphi,x) = -2\int_0^T \varphi'(t)g(t)dx(t) + \int_0^T [\varphi'(t)]^2 g(t)dt. \tag{5}$$

3) there exists a derivative $\varphi^{''}(t)$, which for all $T > 0$ belongs to a compact set in $L_2([0,T])$.

Rewriting the stochastic integral in (5) as an ordinary integral

$$\int_0^T \varphi'(t)g(t)dx(t) = x(T)\varphi^{'}(T)g(T) - \int_0^T x(t)[\varphi'(t)g(t)]^{'}dt$$

we see that there is such a function $\varphi_T(t,x) = \varphi_T^*(t) \in M$, for which the functional from (5) takes the minimal value on M if $\varphi(t) = \varphi_T^*(t)$.

Theorem 1. *Let conditions 1) - 3) be fulfilled and moreover: 4) there is an increasing function $\theta(t)$ such that*

$$P\left\{\lim_{t\longrightarrow\infty}\frac{W(t)}{\theta(t)} = 0\right\} = 1,$$

$$\sup_{\varphi\in M}\int_0^\infty [\varphi'(t)g(t)]^{'}\theta(t)dt < \infty$$

and

$$\sup_t \sup_{\varphi\in M}\frac{|\varphi'(t)g(t)|}{\theta(t)} < \infty.$$

Then for each $\varphi_0 \in M$ and $\epsilon > 0$

$$\lim_{T\to\infty}\mu_{\varphi_0}\left(\left\{x : \int_0^T \left[\frac{d}{dt}(\varphi_0(t) - \varphi_T^*(t))\right] g^2(t)dt > \epsilon\right\}\right) = 0.$$

Proof. The functional (5) can be converted as follows:

$$2\int_0^T \omega'(t)g(t)d\omega(t) + \int_0^T [\omega'(t)]^2 g(t)dt$$

$$-2\int_0^T {\varphi_0}'(t)g(t)d\omega(t) + \int_0^T [{\varphi_0}'(t)]^2 g(t)dt, \tag{6}$$

where $\omega(t) = \varphi(t) - \varphi_0(t)$.

It is evident that the minimum points ω_T^* for the functional (5) and for the functional

$$\widetilde{\Phi}(T,\omega) = -2\int_0^T \omega^{'}(t)g^{'}(t)d\omega(t) + \int_0^T [\omega^{'}(t)]^2 g(t)dt \tag{7}$$

coincide ($\omega_T^* = \varphi_T^* - \varphi_0$).

Suppose that $\widetilde{M_0} = \{\omega : \omega = \varphi - \varphi_0,\ \varphi \in M\}$,

$$S_\epsilon = \left\{\omega : \omega \in \widetilde{M_0},\ \int_0^\infty [\omega'(t)]^2 g^2(t)dt \geq \epsilon\right\}.$$

Then we have

$$\mu_{\varphi_0}\left(\left\{x : \varphi_T^* \in M,\ \int_0^T \left[\frac{d}{dt}(\varphi_T^*(t) - \varphi_0(t))\right]^2 g^2(t)dt \geq \epsilon\right\}\right)$$
$$\leq P\{\inf_{\omega \in S_\epsilon} \widetilde{\Phi}(T,\omega) > 0\}. \tag{8}$$

It is easy to prove that with probability one the functional (7) convergences to $+\infty$ uniformly with respect to $\omega' \in S_\epsilon$ as $T \to \infty$. □

In the linear shell $\widetilde{M_0}$ we introduce the following norm

$$||\omega||_g = \left(\int_0^T [\omega'(t)]^2 g^2(t)dt\right)^{1/2}. \tag{9}$$

According to Theorem 1 $\widetilde{M_0}$ is compact in norm (9). Let on the σ-algebra generated by Borel sets from $\widetilde{M_0}$ in the norm (9) the measure $\theta(da)$ be given. Consider the following Bayesian criterion

$$\varphi_T^*(x) = \frac{\int \varphi \exp\{\int_0^T \varphi'(t)g(t)dt - \frac{1}{2}\int_0^T [\varphi'(t)]^2 g(t)dt\}\theta(da)}{\int \exp\{\int_0^T \varphi'(t)g(t)dt - \frac{1}{2}\int_0^T [\varphi'(t)]^2 g(t)dt\}\theta(da)}. \tag{10}$$

It is not difficult to prove the following theorem.

Theorem 2. *Let M be a convex set satisfying conditions 1) - 4), for all $\epsilon > 0$ one can indicate $\lambda > 1$ and $\epsilon_1 < \epsilon$ such that for all sufficiently large T and $\omega \in \widetilde{M_0}$:*

$$\sup_{||\omega||_g \leq \epsilon_1} \int_0^T [\omega'(t)]^2 g(t)dt \leq \frac{1}{\lambda} \inf_{||\omega||_g > \epsilon} \int_0^T [\omega'(t)]^2 g(t)dt,$$

and a measure $\theta(da)$ such that $\forall \varphi_1 \in M : \theta(\{\varphi : ||\varphi - \varphi_1||_g \leq \epsilon\}) > 0$, then $\epsilon > 0$:

$$\lim_{T\to\infty} P\{||\varphi_T^* - \varphi_0|| > \epsilon\} = 0.$$

R E F E R E N C E S

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