Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 33, 2019

POSITIVE INTEGERS NOT REPRESENTED BY A BINARY QUADRATIC FORM

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Abstract. In this paper the formulae for the average number of representations of positive integers by a genus of positive binary quadratic forms are obtained. This allows us to establish the conditions when it is impossible to represent a positive integer by a binary form.

Keywords and phrases: Binary form, genera of binary forms, number of representations.

AMS subject classification (2010): 11E20, 11E25.

Let $F(\tau; f)$ denote a theta-series of a genus containing a primitive quadratic form f. Siegel [1] proved that if the number of variables of a quadratic form f is $s > 4$, then $F(\tau; f) = E(\tau; t)$, where $E(\tau; f)$ is the Eisenstein series. Later Ramanathan [2] proved that for any primitive quadratic form f with $s \geq 3$ variables (except for zero forms with variables $s = 3$ and zero forms with variables $s = 4$ whose discriminant is a perfect square), there is a function $E(\tau, z; f)$, which he called the Eisenstein-Siegel series and which is regular for any fixed τ when $\operatorname{Im}\tau > 0$ and $\operatorname{Re} z > 2 - \frac{s}{2}$, analytically extendable in a neighbourhood of $z = 0$, and that $F(\tau; f) = E(\tau, z; f)|_{z=0}$.

In [3] we proved that the function $E(\tau, z; f)$ is analytically extendable in a neighborhood of $z = 0$ also in the case where f is any nonzero integral binary quadratic form and that

$$F(\tau; f) = \frac{1}{2}\, E(\tau, z; f)|_{z=0} = 1 + \sum_{n=1}^{\infty} \frac{1}{2}\, \rho(n; f) e^{2\pi i\tau n}. \tag{1}$$

From the results of [3] and [4] we obtain that the function $\rho(n; f)$ is equal to the average number of representations of a natural number by the genus containing this quadratic form and in case of binary forms can be calculated as follows.

Theorem 1. *Let $f = ax^2+bxy+cy^2$ be a primitive positive positive binary quadratic form with discriminant $d = b^2 - 4ac$, $(a, d) = 1$, $\Delta = -\frac{d}{4}$ if $2|b$, $\Delta = -d$ if $2 \nmid b$; $\Delta = r^2\omega$ (ω is a square free number), $n = 2^\alpha m$ $(2 \nmid m)$, $\Delta = 2^\gamma \Delta_1$, $(2 \nmid \Delta_1)$, $p^\ell \| \Delta$, $p^\beta \| n$, $u = \prod\limits_{\substack{p|n\\ p|2\Delta}} p^\beta$,*

then

$$\rho(n; f) = \frac{\pi\chi_2 \prod\limits_{p|\Delta, p>2} \chi_p \sum\limits_{\nu|u} \left(\frac{-\Delta}{\nu}\right)}{\Delta^{\frac{1}{2}} \prod\limits_{p|r, p>2} \left(1 - \left(\frac{-w}{p}\right)\frac{1}{p}\right) L(1, -\omega)}.$$

Here

for $2|d$

$$\begin{aligned}
\chi_2 &= 2^{\frac{\alpha}{2}+2} \ \text{ if } \ 2|\gamma, \ \ 0 \le \alpha \le \gamma - 3, \ \ 2|\alpha, \ \ m \equiv a \,(\operatorname{mod} 8);\\
&= 0 \ \text{ if } \ 2|\gamma, \ \ 0 \le \alpha \le \gamma - 3, \ \ 2|\alpha, \ \ 2|\gamma, \ \ m \not\equiv a \,(\operatorname{mod} 8) \ \text{ or}\\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 \le \alpha \le \gamma - 1, \ \ 2 \nmid \alpha;\\
&= \Big(1 + (-1)^{\frac{1}{2}(m-a)}\Big) 2^{\frac{\alpha}{2}} \ \text{ if } \ 2|\gamma, \ \ \alpha = \gamma - 2;\\
&= \Big(1 + (-1)^{\frac{1}{2}(m-a)}\Big) 2^{\frac{\gamma}{2}} \ \text{ if } \ 2|\gamma, \ \ \alpha \ge \gamma, \ \ 2|\alpha, \ \ \Delta_1 \equiv 1 \,(\operatorname{mod} 4);\\
&= 2^{\frac{\gamma}{2}} \ \text{ if } \ 2|\gamma, \ \ \alpha = \gamma, \ \ \Delta_1 \equiv -1 \,(\operatorname{mod} 4);\\
&= \Big(2 - (-1)^{\frac{1}{4}(\Delta_1+1)}\Big) 2^{\frac{\gamma}{2}} + \Big(1 + (-1)^{\frac{1}{4}(\Delta_1+1)}\Big)(\alpha - \gamma - 2) 2^{\frac{\gamma}{2}-1}\\
&\qquad\qquad\qquad \text{if } \ 2|\gamma, \ \ \alpha > \gamma, \ \ 2|\alpha, \ \ \Delta_1 \equiv -1 \,(\operatorname{mod} 4);\\
&= \Big(1 + (-1)^{\frac{1}{4}(\Delta_1-1)+\frac{1}{2}(m-a)}\Big) 2^{\frac{\gamma}{2}} \ \text{ if } \ 2|\gamma, \ \ \alpha \ge \gamma + 1, \ \ 2 \nmid \alpha,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \Delta_1 \equiv 1 \,(\operatorname{mod} 4);\\
&= \Big(1 + (-1)^{\frac{1}{4}(\Delta_1+1)}\Big)(\alpha - \gamma - 1) 2^{\frac{\gamma}{2}-1} \ \text{ if } \ 2|\gamma, \ \ \alpha \ge \gamma + 1, \ \ 2 \nmid \alpha,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \Delta_1 \equiv -1 \,(\operatorname{mod} 4);\\
&= 2^{\frac{\alpha}{2}+2} \ \text{ if } \ 0 \le \alpha \le \gamma - 3, \ \ 2 \nmid \gamma, \ \ 2|\alpha, \ \ m \equiv a \,(\operatorname{mod} 8);\\
&= 0 \ \text{ if } \ 0 \le \alpha \le \gamma - 3, \ \ 2 \nmid \gamma, \ \ 2|\alpha, \ \ m \not\equiv a \,(\operatorname{mod} 8) \ \text{ or}\\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 \le \alpha \le \gamma - 2, \ \ 2 \nmid \gamma, \ \ 2 \nmid \alpha;\\
&= \Big(1 + (-1)^{\frac{1}{4}(m-a)}\Big) 2^{\frac{1}{2}(\gamma-1)} \ \text{ if } \ 2 \nmid \gamma, \ \ \alpha \ge \gamma - 1, \ \ 2|\alpha,\\
&\qquad\qquad\qquad\qquad\qquad\qquad m \equiv a \,(\operatorname{mod} 4);\\
&= \Big(1 + (-1)^{\frac{1}{4}(m+a)+\frac{1}{2}(m-\Delta_1 a)}\Big) 2^{\frac{1}{2}(\gamma-1)} \ \text{ if } \ 2 \nmid \gamma, \ \ \alpha \ge \gamma - 1, \ \ 2|\alpha,\\
&\qquad\qquad\qquad\qquad\qquad\qquad m \equiv -a \,(\operatorname{mod} 4);\\
&= \Big(1 + (-1)^{\frac{1}{4}(m-\Delta_1 a)}\Big) 2^{\frac{1}{2}(\gamma-1)} \ \text{ if } \ 2 \nmid \gamma, \ \ \alpha \ge \gamma, \ \ 2 \nmid \alpha,\\
&\qquad\qquad\qquad\qquad\qquad\qquad m \equiv \Delta_1 a \,(\operatorname{mod} 4);\\
&= \Big(1 + (-1)^{\frac{1}{4}(m+\Delta_1 a)+\frac{1}{2}(m-a)}\Big) 2^{\frac{1}{2}(\gamma-1)} \ \text{ if } \ 2 \nmid \gamma, \ \ \alpha \ge \gamma, \ \ 2 \nmid \alpha,\\
&\qquad\qquad\qquad\qquad\qquad\qquad m \equiv -\Delta_1 a \,(\operatorname{mod} 4);
\end{aligned}$$

for $2 \nmid d$,

$$\begin{aligned}
\chi_2 &= 3 \ \text{ if } \ 2 \nmid \alpha, \ \ \Delta \equiv 3 \,(\operatorname{mod} 8);\\
&= 0 \ \text{ if } \ 2 \nmid \alpha, \ \ \Delta \equiv 3 \,(\operatorname{mod} 8);\\
&= \alpha + 1 \ \text{ if } \ \Delta \equiv 7 \,(\operatorname{mod} 8);\\
\chi_p &= \left(1 + \left(\frac{p^{-\beta} na}{p}\right)\right) p^{\frac{1}{2}\beta} \ \text{ if } \ \ell \ge \beta + 1, \ \ 2|\beta;\\
&= \left(1 - \left(\frac{-p^{-\ell}\Delta}{p}\right)\frac{1}{p}\right)\left\{1 + \left(1 + \left(\frac{-p^{-\ell}\Delta}{p}\right)\right)\frac{\beta - \ell}{2}\right\} p^{\frac{1}{2}\ell} \ \text{ if}
\end{aligned}$$

$$\ell \le \beta,\ \ 2|\ell,\ \ 2|\beta;$$

$$= \Big(1 - \Big(\frac{-p^{-\ell}\Delta}{p}\Big)\frac{1}{p}\Big)\Big(1 + \Big(\frac{-p^{-\ell}\Delta}{p}\Big)\Big)\frac{\beta-\ell+1}{2}p^{\frac{1}{2}\ell} \ \text{ if}$$

$$\ell \le \beta,\ \ 2|\ell,\ \ 2\nmid\beta;$$

$$= \Big(1 + \Big(\frac{p^{-\ell}\Delta}{p}\Big)^{\beta+1}\Big(\frac{p^{-(\beta+\ell)}na\Delta}{p}\Big)\Big)p^{\frac{1}{2}(\ell-1)} \ \text{ if }\ \ell \le \beta,\ \ 2\nmid\ell;$$

$$= 0 \ \text{ if }\ \ell \ge \beta+1,\ \ 2\nmid\beta.$$

The values of $L(1;-\omega)$ can be calculated by the formulas of [5] (Theorem 15).

It follows from (1) that half of "the sum of a generalized singular series" that corresponds to a binary quadratic form f is equal to the average number of representations of natural numbers by the genus containing this quadratic form. Therefore, if $\rho(n;t)=0$, then n is an integer, not represented by a quadratic form, belonging to the genus of f. By Theorem 1 we have the following result.

Theorem 2. *Let $f = ax^2+bxy+cy^2$ be a primitive positive quadratic form with discriminant $d=b^2-4ac$, $(a,d)=1$, $\Delta=-\frac{d}{4}$ if $2|b$, $\Delta=-d$ if $2\nmid b$, $n=2^\alpha m$ $(2\nmid m)$, $\Delta=2^\gamma\Delta_1(2\nmid\Delta_1)$ $p^\ell\|\Delta$, $p^\beta\|n$, then $r(n;f)=0$, if one of these following conditions holds:*

1) $2|d$, $0\le\alpha\le\gamma-3$, $2|\alpha$, $m\not\equiv a(\text{mod }8)$;
2) $2|d$, $0\le\alpha\le\gamma-1$, $2\nmid\alpha$;
3) $2|d$, $2|\alpha$, $\alpha=\gamma-2$, $m\equiv(a+2)(\text{mod }4)$;
4) $2|d$, $2|\alpha$, $2|\gamma$, $\alpha\ge\gamma$, $\Delta_1\equiv1(\text{mod }4)$;
5) $2|d$, $2\nmid\alpha$, $2|\gamma$, $\alpha>\gamma$, $\Delta_1\equiv1(\text{mod }8)$, $m\equiv(a+2)(\text{mod }4)$;
6) $2|d$, $2\nmid\alpha$, $2|\gamma$, $\alpha>\gamma$, $\Delta_1\equiv5(\text{mod }8)$, $m\equiv a(\text{mod }4)$;
7) $2|d$, $2\nmid\alpha$, $2|\gamma$, $\alpha>\gamma$, $\Delta_1\equiv3(\text{mod }8)$;
8) $2|d$, $2|\alpha$, $2\nmid\gamma$, $\alpha\ge\gamma-1$, $m\equiv(a+4)(\text{mod }8)$;
9) $2|d$, $2|\alpha$, $2\nmid\gamma$, $\alpha\ge\gamma-1$, $m\equiv-a(\text{mod }8)$, $\Delta_1\equiv(\text{mod }4)$;
10) $2|d$, $2|\alpha$, $2\nmid\gamma$, $\alpha\ge\gamma-1$, $m\equiv(-a+4)(\text{mod }8)$, $\Delta_1\equiv3(\text{mod }4)$;
11) $2|d$, $2\nmid\alpha$, $2\nmid\gamma$, $\alpha\ge\gamma$, $m\equiv(\Delta_1a+4)(\text{mod }8)$;
12) $2|d$, $2\nmid\alpha$, $2\nmid\gamma$, $m\equiv(-\Delta_1a+4)(\text{mod }8)$, $m\equiv a(\text{mod }4)$;
13) $2|d$, $2\nmid\alpha$, $2\nmid\gamma$, $\alpha\ge\gamma$, $m\equiv-\Delta_1a(\text{mod }8)$, $m\equiv(a+2)(\text{mod }4)$;
14) $2\nmid d$, $\Delta\equiv3(\text{mod }8)$, $2\nmid\alpha$;
15) $\Big(\frac{-p^{-\ell}\Delta}{p}\Big)=-1$, $\ell\le\beta$, $2|\ell$, $2\nmid\beta$;
16) $\Big(\frac{-p^{-\beta}na}{p}\Big)=-1$, $\ell\ge\beta+1$, $2|\beta$;
17) $\Big(\frac{p^{-\ell}\Delta}{p}\Big)^{\beta+1}\Big(\frac{p^{-(\beta+\ell)}na\Delta}{p}\Big)=-1$, $\ell\le\beta$, $2\nmid\ell$, $\ell\ge\beta+1$, $2\nmid\beta$.

Some special cases of Theorem 2 are given in [6].

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Received 31.05.2019; revised 18.11.2019; accepted 21.12.2019.

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