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STRUCTURE OF THE "d_m-KNIVES" AND PROCESS OF CUTTING OF GML_m^n OR GRT_m^n BODIES

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Abstract. This article discusses the design and geometric properties of the d_m-knives.

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In our previous works [1-2], we have shown the close connection of the process of full cutting of GML_m^n bodies with one knife ($gcd(m,n)=1$) and a similar process of cutting the of GRT_m^n bodies with a $d_m = m$ -knife. All reasoning is based on the analytical presentation (1) of the bodies and corresponding "knives" (i.e. slit-surfaces). One form of the analytical representation, defining these geometric objects, is formula (1) in [1].

$$\begin{array}{l} X(x,z,\theta) = [R(\theta,t) + p(x,z)\cos(\mu g(\theta))]\cos(\theta), \\ Y(x,z,\theta) = [R(\theta,t) + p(x,z)\cos(\mu g(\theta))]\sin(\theta), \\ Z(x,z,\theta) = K(\theta) + p(x,z)\sin(\mu g(\theta)), \end{array} \tag{1}$$

where the main difference between the bodies is the structure of the function $K(\theta)$; for GML_m^n bodies $K(\theta)$ is always 0 or a 2π-periodic function, and for GRT_m^n bodies $K(\theta) = const \neq 0$ or a non-self-intersecting function. The d_m-knife [1] is defined as:

Definition. Slit surfaces may be represented by (1) where the function $p(x,z)$ is a $d_m \equiv m$ -knife, in other words, it is a construction with m straight lines and has the form:

$$\begin{array}{c} x_i \cdot \sin(\alpha + \frac{2i\pi}{m}) + z_i \cdot \cos(\alpha + \frac{2i\pi}{m}) + \delta = 0, \\ i = 0,1,...,m-1, \quad -\frac{\pi}{m} \leq \alpha \leq \frac{\pi}{m}, \end{array} \tag{2}$$

where α is the rotation parameter and δ is a zoom parameter. It should be specially noted that the important case is when d_m is a divisor of the number m, but in this article we will always keep in mind that $d_m \equiv m$.

The connecting link, from a mathematical point of view, of these two actions was the trace left by these "knives" on the radial cross sections of GML_m^n and normal cross section of GRT_m^n bodies - these traces are identical in both cases! For this reason, it turned out to be very important to study in more detail the construction and internal properties of the d_m-knife itself.

Some of the important questions are: How many intersections result from d_m-knife? How many and which bounded closed and open unlimited shapes-figures result with the d_m-knife, and which of them are equal to each other?

Proposition 1. *Case (**m is odd**): If $\delta \equiv 0$ and $m = 2k+1$ is odd then the corresponding d_m-knife consists of m lines, has a single point of intersection, the origin, and forms $2m$ pieces of equal and classic angles (open unlimited shapes) on the plane (example when $m = 9$ see Figure 1a).*

*Case (**m is even**): If $\delta \equiv 0$ and $m = 2k$ is even then the corresponding d_m-knife consists of k lines, has a single point of intersection, the origin, and forms m pieces of equal and classic angles (open unlimited shapes) on the plane (example when $m = 12$ see Figure 1b).*

Proposition 2. *Case (**m is odd**): If $\delta \neq 0$ and $m = 2k+1$ is odd then the corresponding d_m-knife :*

•1. *Consists of m lines, has $k(2k+1)$ points of intersection, which are located on the k - different circles with radius $r_i = \delta \cdot \sqrt{2(1 - \cos(\frac{2i\pi}{m}))} \cdot (\sin(\frac{2i\pi}{m}))^{-1}$, $i = 1, 2, ..., k$ and on each "circle-i" there are $2k+1$ intersection points, which are equally spaced;*

•2. *Consists of $(k-1)(2k+1)+1$ pieces of bounded closed figures and $2(2k+1)$ angles (open unlimited) on the plane; more precisely, these bounded closed figures can always be sorted as follows:*

$\star a$. One regular m-gon whose vertices, the points of intersection of the subsequent lines, are located on circle-1 with a minimum radius r_1 and with an internal angle equal to $\frac{(2k-1)\pi}{2k+1}$;

$\star b$. First layer: m -pieces of equal triangles, two vertices of which are located on a circle-1 and internal angles equal to $\frac{2\pi}{2k+1}$ and a third is on a circle-2 and equal to $\frac{(2k-3)\pi}{2k+1}$;

$\star c$. Second layer:. m-pieces of equal quadrangles, one vertex of which is located on a circle-1 and equal to $\frac{(2k-1)\pi}{2k+1}$, two others on a circle-2 equals to $\frac{4\pi}{2k+1}$ and the fourth one on a circle-3 equal to $\frac{(2k-5)\pi}{2k+1}$;

$\star d$. Layers number $j = 2, 3, .., k-1$ have a structure similar to the second layer and consist of m pieces of equal quadrilaterals, one vertex of which is located on circle-$(j-1)$ with angle $\frac{(2k-2j+3)\pi}{2k+1}$, two vertices on circle-j and with angles are $\frac{2j\pi}{2k+1}$ and the fourth is a vertex on circle-$(j+1)$ and with angle $\frac{(2k-2j+1)\pi}{2k+1}$ (examples in Fig 1c and Fig 2a).

•3. Open unlimited angles are divided into two groups:

$\star a$. m identical pieces with classic infinite angles with vertices situated on the circle-k with a solution $\frac{\pi}{2k+1}$;

$\star b$. m pieces with equal "distinct angles", which consists of one corner with a solution of $\frac{3\pi}{2k+1}$ and two angles equal to $\frac{2k\pi}{2k+1}$ for every k except $k = 1$; if $k = 1$ then this distinct angle consists of only two angles $\frac{2\pi}{3}$ (see Figure 1c).

Proposition 3. Case (**m is even**): If $\delta \neq 0$ and $m = 2k$ is even then the corresponding d_m-knife:

•1. Consists of m lines, has $2k(k-1)$ points of intersection, which are located on the $k-1$ - different circles with radius $r_i = \delta \cdot \sqrt{2(1 - \cos(\frac{i\pi}{k}))} \cdot (\sin(\frac{i\pi}{k}))^{-1}$, $i = 1, 2, ..., k-1$ and on each "circle-i" there are $2k$ intersection points, which are equally spaced;

•2. Consists of $2k(k-2)+1$ pieces of bounded closed figures and $4k$ angles (open unlimited) on the plane; more precisely, these bounded closed figures can always be sorted as follows:

⋆*a*. One regular m-gon whose vertices, the points of intersection of the subsequent lines, are located on circle-1 with a minimum radius r_1 and with an internal angle equal to $\frac{(k-1)\pi}{k}$;

⋆*b*. First layer: m -pieces of equal triangles, two vertices of which are located on a circle-1 and internal angles equal to $\frac{\pi}{k}$ and a third is on a circle-2 and equal to $\frac{(k-2)\pi}{k}$;

⋆*c*. Second layer:. m-pieces of equal quadrangles, one vertex of which is located on a circle-1 and equal to $\frac{(k-1)\pi}{k}$, two others on a circle-2 equals to $\frac{2\pi}{k}$ and the fourth one on a circle-3 equal to $\frac{(k-3)\pi}{k}$;

⋆*d*. Layers number $j = 2, 3, .., k-2$ have a structure similar to the second layer and consist of m pieces of equal quadrilaterals, one vertex of which is located on circle-$(j-1)$ with angle $\frac{(k-j+1)\pi}{k}$, two vertices on circle-j and with angles are $\frac{j\pi}{k}$ and the fourth is a vertex on circle-$(j+1)$ and with angle $\frac{(k-j-1)\pi}{k}$ (examples in Figure 1d and Figure 2b).

•3. Open unlimited angles are divided into two groups:

⋆*a*. m identical pieces identical with classic infinite angles with vertices situated on the circle-$(k-1)$ with a solution $\frac{\pi}{k}$;

⋆*b*. m pieces with equal “distinct angles”, which consists of one corner with a solution of $\frac{2\pi}{k}$ and two angles equal to $\frac{(k-1)\pi}{k}$ for every k except $k=2$, if $k=2$ then this distinct angle consists of only two angles $\frac{\pi}{2}$ (see Figure 1d).

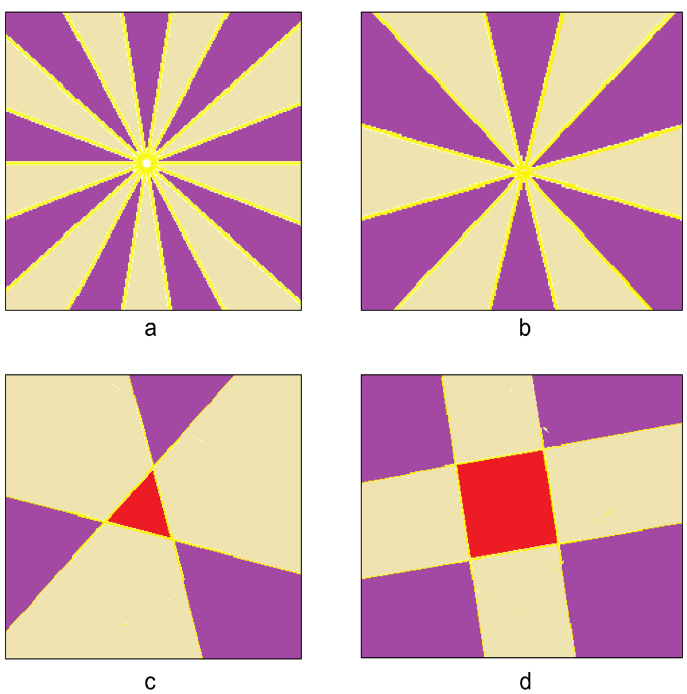
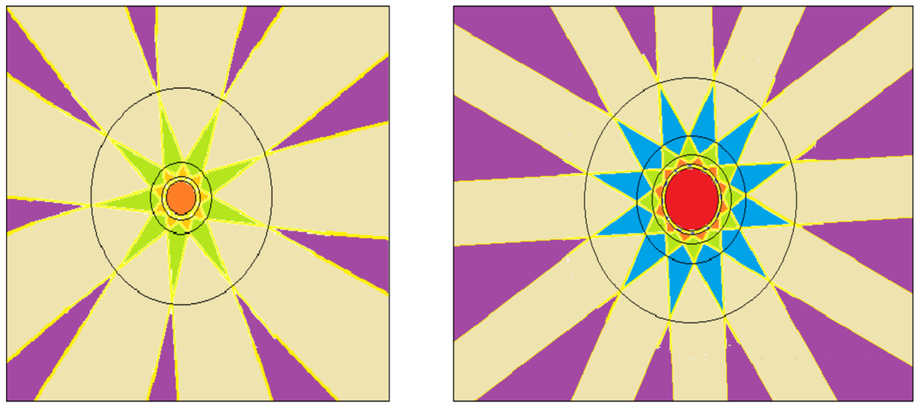


Figure 1:



a b

Figure 2:

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