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THE ALGORITHM TO CONSTRUCT THE EIGENFUNCTIONS OF THE MULTI-VELOCITY TRANSPORT THEORY

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Abstract. We present the algorithm to construct the eigenfunctions of the characteristic equation of the multi-velocity transport theory by the Legendre polynomials.

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The work deals with the questions to construct the eigenfunctions of the characteristic equation of the multi-velocity transport theory by the Legendre polynomials.

The characteristic equation of the transport theory occupies a central position in the mathematical theory of the passage of radiation through matter. This theory was first developed as a branch of astrophysics, and was initially applied to problems of the passage of light rays through stellar atmospheres. Its sphere of application has now considerably broadened; it forms the basis of modern reactor theory and of the theory of protection against penetrating radiations, and with the of cosmic investigations is becoming increasingly widely used in atmospheric optic.

We consider the Boltzman linear integro differential equation having the following form [1]

$$\mu\frac{\partial\psi}{\partial x}+\psi(x,\mu,E)=\int_{E_1}^{E_2}\int_{-1}^{+1}K\psi d\mu' dE',$$

$$x\in(-\infty,+\infty),\quad \mu\in(-1,+1),\quad E\in[E_1,E_2],$$

where

$$K(\mu,E;\mu',E')=\sum_{n=0}^{N}\frac{2n+1}{2}K_n(E,E')P_n(\mu)P_n(\mu'),$$

$K_n(E,E')$ are continuous functions and $P_n(\mu)$ is the Lejendre polynomial of order n.

The general solution of the considered equation which is continuously, differentiable with respect to x and satisfying H^* conditions [2] with respect to μ can be writen by the formula [see 3-4]

$$\psi(x,\mu,E)=\sum_{k}c_k\exp(-x/\nu_k)\varphi_{\nu_k}(\mu,E)$$

$$+\int_{E_1}^{E_2}\int_{-1}^{+1}\exp(-x/\nu)\varphi_{\nu,(\zeta)}(\mu,E)u(\nu,\zeta)d\zeta d\nu,$$

$$x \in (-\infty, +\infty), \quad \mu \in (-1, +1), \quad E \in [E_1, E_2],$$

where $\{\varphi_{\nu_k}\}$ is regular (we suppose that the set of eigenvalues is finite) and

$$\varphi_{\nu,(\zeta)}(\mu, E) = \frac{\nu R(\nu, \zeta; \mu, E)}{\nu - \mu}$$

$$+\left(\delta(\zeta - E) - \int_{-1}^{+1} \frac{\nu R(\nu, \zeta; \mu', E)}{\nu - \mu} d\mu'\right)\delta(\nu - \mu)$$

the singular eigenfunction of the characteristic equation

$$(\nu - \mu)\varphi(\mu, E) = \nu \int_{E_1}^{E_2} \int_{-1}^{+1} K\varphi d\mu' dE',$$

$$\nu, \mu \in (-1, +1), \quad E \in [E_1, E_2]$$

respectively. Here $R(\nu, \zeta; \mu, E)$ is the solution of the following equation

$$R(\nu, \zeta; \mu, E) = K(\mu, E; \nu, \zeta) \tag{1}$$

$$+\nu \int_{E_1}^{E_2} \int_{-1}^{+1} \frac{K(\mu, E; \mu', E') - K(\mu, E; \nu, E')}{\nu - \mu'} R(\nu, \zeta; \mu', E' d\mu' dE'),$$

$$\nu,\ \mu \in (-1, +1).$$

Using the following property of the Legandre polynomials

$$P_n(\mu) = \nu \frac{P_n(\mu) - P_n(\nu)}{\nu - \mu}$$

$$-\frac{n+1}{2n+1} \frac{P_{n+1}(\mu) - P_{n+1}(\nu)}{\nu - \mu}$$

$$-\frac{n}{2n+1} \frac{P_{n-1}(\mu) - P_{n-1}(\nu)}{\nu - \mu}$$

we can prove the following lemma.

Lemma. *For every value of parameter $\nu \in (-1, +1)$ the homogeneous equation*

$$R_0(\nu, \zeta; \mu, E)$$

$$= \nu \int_{E_1}^{E_2} \int_{-1}^{+1} \frac{K(\mu, E; \mu', E') - K(\mu, E; \nu, E')}{\nu - \mu'} R_0(\nu, \zeta; \mu', E') d\mu' dE',$$

$$\nu, \mu \in (-1, +1)$$

admits only zero zolution.

Also, using the following property of the Legandre polynomials

$$P_n(\mu) = \nu\left(\frac{P_n(\mu) - P_n(\nu)}{\nu - \mu} + P_n(\nu)\right)$$

$$-\frac{n+1}{2n+1}\left(\frac{P_{n+1}(\mu) - P_{n+1}(\nu)}{\nu - \mu} + P_{n+1}(\nu)\right)$$

$$-\frac{n}{2n+1}\left(\frac{P_{n-1}(\mu) - P_{n-1}(\nu)}{\nu - \mu} + P_{n-1}(\nu)\right)$$

of the Legandre polynomials for the $R(\nu, \zeta; \mu, E)$ we can prove the following theorem.

Theorem. *The integral equation (1) admits continuously, unique solution representing by the formula*

$$R(\nu, \zeta; \mu, E) = \sum_{n=0}^{N} \frac{2n+1}{2} P_n(\mu) h_n(\nu, \zeta, E),$$

where

$$\nu h_n(\nu, \zeta, E) - \frac{n+1}{2n+1} h_{n+1}(\nu, \zeta, E) - \frac{n}{2n+1} h_{n-1}(\nu, \zeta, E)$$

$$-\nu \int_{E_1}^{E_2} K_i(E, E') h_i(\nu, \zeta, E') dE' = P_i(\nu) K_n(E, \zeta),$$

$h_{-1}(\nu, \zeta, E) = h_0(\nu, \zeta, E) = 0; \;\; n = 0, 1, \ldots .$

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