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# THE SPACES OF SPHERICAL POLYNOMIALS AND GENERALIZED THETA-SERIES

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**Abstract**. The dimension of the space of theta-series and of vector space of generalized theta-series corresponding to some nondiagonal ternary quadratic forms are established and the bases of these spaces constructed.

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## 1 Introduction.

Let

$$Q(X) = Q(x_1, x_2, \dots, x_r) = \sum_{1\le i\le j\le r} b_{ij} x_i x_j$$

be an integer positive definite quadratic form of $r$ variables and let $A = (a_{ij})$ be the symmetric $r \times r$ matrix of the quadratic form $Q(X)$, where $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, for $i < j$. If $X = (x_1 \dots x_r)^T$ denotes a column matrix and $X^T$ its transpose, then $Q(X) = \frac{1}{2} X^T A X$. Let $A_{ij}$ denote the cofactor to the element $a_{ij}$ in $A$ and $a^*_{ij}$ is the element of the inverse matrix $A^{-1}$.

A homogeneous polynomial $P(X) = P(x_1, \cdots, x_r)$ of degree $\nu$ with complex coefficients, satisfying the condition

$$\sum_{1\le i,j\le r} a^*_{ij} \Big(\frac{\partial^2 P}{\partial x_i \partial x_j}\Big) = 0 \qquad (1)$$

is called a spherical polynomial of order $\nu$ with respect to $Q(X)$ (see [2]).

Let $\mathcal{P}(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order $\nu$ with respect to $Q(X)$. Hecke [3] calculated the dimension of the space $\mathcal{P}(\nu, Q)$ and showed that

$$\dim \mathcal{P}(\nu, Q) = \binom{\nu + r - 1}{r-1} - \binom{\nu + r - 3}{r-1}.$$

He formed a basis of the space of spherical polynomials of second order ($\nu = 2$) with respect to $Q(X)$. Lomadze [4] constructed a basis of the space of spherical polynomials of fourth order ($\nu = 4$) with respect to $Q(X)$. In the next section a basis of the space $\mathcal{P}(\nu, Q)$ for $\nu = 4$ is constructed in a simpler way.

Let

$$\vartheta(\tau, P, Q) = \sum_{n\in\mathbb{Z}^r} P(n) z^{Q(n)}, \qquad z = e^{2\pi i \tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im}\tau > 0$$

be the corresponding generalized $r$-fold theta-series.

Let $T(\nu, Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.

$$T(\nu, Q) = \{\vartheta(\tau, P, Q) \; : \; P \in \mathcal{P}(\nu, Q)\}.$$

Gooding [2] calculated the dimension of the vector space $T(\nu, Q)$ for the reduced binary quadratic form $Q$ and obtained an upper bound of the dimension of the space $T(\nu, Q)$ for some diagonal quadratic form of $r$ variables $\dim T(\nu, Q) \leq \binom{\frac{\nu}{2}+r-2}{r-2}$. In [5, 6], the upper bounds for the dimensions of the spaces $T(\nu, Q)$ for some quadratic forms are established, in a number of cases the dimensions are calculated and the bases of these spaces are formed. Gaigalas [1] obtained the upper bounds for the dimensions of the spaces $T(4, Q)$ and $T(6, Q)$ for some diagonal quadratic forms and presented the upper bounds of the dimensions of the spaces $T(\nu, Q)$ for some diagonal quadratic forms of six variables.

In this paper the dimensions of the spaces $\mathcal{P}(\nu, Q)$ and $T(4, Q)$ for nondiagonal ternary quadratic form are obtained and a bases of this spaces are constructed.

**2 The basis of the space $\mathcal{P}(\nu, Q)$.** Let

$$P(X) = P(x_1, x_2, x_3, \cdots x_r) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} \cdots \sum_{l=0}^{m} a_{kij\cdots l} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} \cdots x_r^{l}$$

be a spherical function of order $\nu$ with respect to the positive quadratic form $Q(x_1, x_2, x_3, \cdots, x_r)$ of $r$ variables and let $L = (a_{000\cdots 0}, a_{100\cdots 0}, a_{110\cdots 0}, a_{111\cdots 0}, \ldots, a_{\nu\nu\nu\cdots\nu})^T$ be the column vector, where $a_{kij\cdots l}$ $(\nu \geq k \geq i \geq j \geq \cdots \geq l \geq 0)$ are the coefficients of the polynomial $P(X)$.

The condition (1) in matrix notation has the following form $S \cdot L = 0$, where the matrix $S$ has the form

$$S = \begin{pmatrix} A_{11}\nu(\nu-1) & 2A_{12}(\nu-1) & 2A_{13}(\nu-1) & 2A_{14}(\nu-1) & \ldots & \ldots & \ldots & 0 \\ 0 & A_{11}(\nu-1)(\nu-2) & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & A_{11}(\nu-1)(\nu-2) & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & A_{11}(\nu-1)(\nu-2) & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 2A_{11} & \ldots & A_{rr}(\nu-1)\nu \end{pmatrix}$$

and is $\binom{\nu+r-3}{r-1} \times \binom{\nu+r-1}{r-1}$ matrix.

The construction of the matrix $S$ and the spherical polynomials for $\nu = 4$ and $r = 3$ are considered in the next section.

**3 On the dimension of $T(4, Q)$ for ternary quadratic form.** Consider the nondiagonal quadratic form $Q_1(x_1, x_2, x_3) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{12}x_1x_2$. We have $|A| = detA = 2b_{33}(4b_{11}b_{22} - b_{12}^2)$, $A_{11} = 4b_{22}b_{33}$, $A_{12} = -2b_{12}b_{33}$, $A_{22} = 4b_{11}b_{33}$, $A_{13} = A_{23} = 0$, $A_{33} = 4b_{11}b_{22} - b_{12}^2$.

Let

$$P(X) = P(x_1, x_2, x_3) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} a_{ki} x_1^{\nu-k} x_2^{k-i} x_3^{i}$$

be a spherical function of order $\nu$ with respect to the ternary quadratic form $Q_1(x_1, x_2, x_3)$ and let

$$L = (a_{00}\ a_{10}\ a_{11}\ a_{20}\ a_{21}\ a_{22}\ a_{30}\ \ldots\ a_{\nu\nu})^T$$

be a column vector, where $a_{ki}$ $(\nu \geq k \geq i \geq 0)$ are the coefficients of the polynomial $P(x_1, x_2, x_3)$.

In the matrix equation $S \cdot L = 0$, for $\nu = 4$ the matrix $S$ has the following form

$$S = \begin{pmatrix} 12A_{11} & 6A_{12} & 0 & 2A_{22} & 0 & 2A_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6A_{11} & 0 & 8A_{12} & 0 & 0 & 6A_{22} & 0 & 2A_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6A_{11} & 0 & 4A_{12} & 0 & 0 & 2A_{22} & 0 & 6A_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2A_{11} & 0 & 0 & 6A_{12} & 0 & 0 & 0 & 12A_{22} & 0 & 2A_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2A_{11} & 0 & 0 & 4A_{12} & 0 & 0 & 0 & 6A_{22} & 0 & 6A_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2A_{11} & 0 & 0 & 2A_{12} & 0 & 0 & 0 & 2A_{22} & 0 & 12A_{33} \end{pmatrix}.$$

Consider all possible polynomials $P_{ki}$, with even indices $i$ and $k = \nu - 1, \nu$; their number is 5 for $\nu = 4$:

$$\begin{aligned}
P_{30} &= \frac{b_{12}(b_{12}^2 - 2b_{11}b_{22})}{4b_{22}^3} x_1^4 + \frac{b_{12}^2 - b_{11}b_{22}}{b_{22}^2} x_1^3 x_2 + \frac{3b_{12}}{2b_{22}} x_1^2 x_2^2 + x_1 x_2^3, \\
P_{32} &= \frac{b_{12}(b_{12}^2 - 4b_{11}b_{22})}{24b_{22}^2 b_{33}} x_1^4 + \frac{b_{12}^2 - 4b_{11}b_{22}}{12b_{22}b_{33}} x_1^3 x_2 + \frac{b_{12}}{2b_{22}} x_1^2 x_3^2 + x_1 x_2 x_3^2, \\
P_{40} &= -\frac{b_{11}(b_{12}^2 - b_{11}b_{22})}{b_{22}^3} x_1^4 - \frac{4b_{11}b_{12}}{b_{22}^2} x_1^3 x_2 - \frac{6b_{11}}{b_{22}} x_1^2 x_2^2 + x_2^4, \\
P_{42} &= \frac{(b_{12}^2 - 4b_{11}b_{22})(b_{12}^2 - 2b_{11}b_{22})}{24b_{22}^3 b_{33}} x_1^4 + \frac{b_{12}(b_{12}^2 - 4b_{11}b_{22})}{6b_{22}^2 b_{33}} x_1^3 x_2 \\
&+ \frac{b_{12}^2 - 4b_{11}b_{22}}{4b_{22}b_{33}} x_1^2 x_2^2 - \frac{b_{11}}{b_{22}} x_1^2 x_3^2 + x_2^2 x_3^2, \\
P_{44} &= \frac{(b_{12}^2 - 4b_{11}b_{22})^2}{16b_{22}^2 b_{33}^2} x_1^4 + \frac{3(b_{12}^2 - 4b_{11}b_{22})}{2b_{22}b_{33}} x_1^2 x_3^2 + x_3^4.
\end{aligned}$$

Now we construct the corresponding generalized theta-series:

$$\begin{aligned}
&\vartheta(\tau, P_{30}, Q_1) = \sum_{n=1}^{\infty} \Big( \sum_{Q_1(x)=n} P_{30}(x) \Big) z^n = \frac{b_{12}(b_{12}^2 - 2b_{11}b_{22})}{2b_{22}^3} z^{b_{11}} + \cdots + 0z^{b_{22}} \\
&+ \cdots + 0z^{b_{33}} + \cdots + \frac{b_{12}(b_{12}^2 - 2b_{11}b_{22})}{b_{22}^3} z^{b_{11}+b_{33}} + \cdots + 0z^{b_{22}+b_{33}} + \ldots, \\
&\vartheta(\tau, P_{32}, Q_1) = \sum_{n=1}^{\infty} \Big( \sum_{Q_1(x)=n} P_{32}(x) \Big) z^n = \frac{b_{12}(b_{12}^2 - 4b_{11}b_{22})}{12b_{22}^2 b_{33}} z^{b_{11}} + \cdots + 0z^{b_{22}} \\
&+ \cdots + 0z^{b_{33}} + \cdots + \Big( \frac{b_{12}(b_{12}^2 - 4b_{11}b_{22})}{6b_{22}^2 b_{33}} + \frac{2b_{12}}{b_{22}} \Big) z^{b_{11}+b_{33}} + \cdots + 0z^{b_{22}+b_{33}} + \ldots,
\end{aligned}$$

$$\vartheta(\tau,P_{40},Q_1)=\sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}P_{40}(x)\Big)z^n=-\frac{2b_{11}(b_{12}^2-b_{11}b_{22})}{b_{22}^3}z^{b_{11}}+\cdots+2z^{b_{22}}$$
$$+\cdots+0z^{b_{33}}+\cdots-\frac{4b_{11}(b_{12}^2-b_{11}b_{22})}{b_{22}^3}z^{b_{11}+b_{33}}+\cdots+4z^{b_{22}+b_{33}}+\ldots,$$
$$\vartheta(\tau,P_{42},Q_1)=\sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}P_{42}(x)\Big)z^n=\frac{(b_{12}^2-4b_{11}b_{22})(b_{12}^2-2b_{11}b_{22})}{12b_{22}^3b_{33}}z^{b_{11}}+\cdots+0z^{b_{22}}$$
$$+\cdots+0z^{b_{33}}+\cdots+\Big(\frac{(b_{12}^2-4b_{11}b_{22})(b_{12}^2-2b_{11}b_{22})}{6b_{22}^3b_{33}}-\frac{4b_{11}}{b_{22}}\Big)z^{b_{11}+b_{33}}+\cdots+4z^{b_{22}+b_{33}}+\ldots,$$
$$\vartheta(\tau,P_{44},Q_1)=\sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}P_{44}(x)\Big)z^n=\frac{(b_{12}^2-4b_{11}b_{22})^2}{8b_{22}^2b_{33}^2}z^{b_{11}}+\cdots+0z^{b_{22}}$$
$$+\cdots+2z^{b_{33}}+\cdots+4\Big(\frac{(b_{12}^2-4b_{11}b_{22})^2}{16b_{22}^2b_{33}^2}+\frac{3(b_{12}^2-4b_{11}b_{22})}{2b_{22}b_{33}}+1\Big)z^{b_{11}+b_{33}}+\cdots+4z^{b_{22}+b_{33}}+\ldots.$$

These generalized theta-series are linearly independent since the determinant of the fifth order constructed from the coefficients of these theta-series is not equal to zero and $\dim T(4,Q_1)\le\frac{(r-1)(r+2)}{2}=5$. Hence these theta-series form the basis of the space $T(4,Q_1)$. The following theorem is valid.

**Theorem.** *Let $Q_1(X)$ be the nondiagonal ternary quadratic form, given by $Q_1(X)=b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+b_{12}x_1x_2$, then $\dim T(4,Q_1)=5$ and the generalized theta-series with spherical polynomials $P_{ki}$ ($k=3$ or $4$; $i$ is even):*

$$\vartheta(\tau,P_{30},Q_1);\vartheta(\tau,P_{32},Q_1);\vartheta(\tau,P_{40},Q_1);\vartheta(\tau,P_{42},Q_1);\vartheta(\tau,P_{44},Q_1)$$

*form the basis of the space $T(4,Q_1)$.*

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