Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 33, 2019

THE ACCURACY OF THE FINITE DIFFERENCE SCHEME FOR A NONLINEAR DYNAMIC BEAM PROBLEM

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Abstract. The paper deals with the boundary value problem for a system of nonlinear integro-differential equations modeling the dynamic state of the Timoshenko beam. To approximate the solution with respect to the time variable an implicit difference scheme is used, the error of which is estimated.

Keywords and phrases: Timoshenko beam model, difference scheme, error estimate.

AMS subject classification (2010): 35L50, 74H15, 74S20.

1 Statement of the problem. Let us consider the system of equations with respect to functions $u(x,t)$, $v(x,t)$, $f(x,t)$, $\varphi(x,t)$, $\psi(x,t)$ as in [3]

$$\begin{gathered} u_t=(cd-a+b\int_0^1 v^2dx)v_x-cd\varphi\,,\\ v_t=u_x,\quad f_t=c\varphi_x-c^2d(\psi-v)\,,\\ \varphi_t=f_x,\quad \psi_t=f\,, \end{gathered} \tag{1}$$

$0<x<1,\quad 0<t\le T$, $a,\ b,\ c,\ d>0$, $cd-a>0$, with the initial and boundary conditions

$$\begin{gathered} u(x,0)=w^{(1)}(x),\quad v(x,0)=w^{(2)}(x),\quad f(x,0)=\psi^{(1)}(x),\\ \varphi(x,0)=\psi^{(2)}_x(x),\quad \psi(x,0)=\psi^{(2)}(x),\\ u(0,t)=u(1,t)=0,\quad f(0,t)=f(1,t)=0\,,\\ 0\le x\le 1,\quad 0\le t\le T\,, \end{gathered} \tag{2}$$

where $w^{(1)}(x)$, $w^{(2)}(x)$, $\psi^{(1)}(x)$, $\psi^{(2)}(x)$ are the known functions.

A system of two second order differential equations which in the Timoshenko theory describes the dynamic behavior of the beam [6], can be reduced to (1), (2) problem. Numerical methods for the Timoshenko nonlinear beam systems are studied in [1]-[5].

2 Difference scheme. We approximate the solution of problem (1), (2) with respect to the spatial variable by means of the finite element method using the spatial step $h=1/N$ and N+1 piecewise linear finite element functions. The resulting Cauchy problem for a nonlinear system of ordinary differential equations with respect to the vector-functions $\boldsymbol{u}_h(t),\boldsymbol{f}_h(t)\in R^{N-1}$, $\boldsymbol{v}_h(t),\boldsymbol{\varphi}_h(t),\boldsymbol{\psi}_h(t)\in R^{N+1}$

$$M\frac{d\boldsymbol{u}_h}{dt}=(cd-a+bh\boldsymbol{v}'_hK\boldsymbol{v}_h)Q\boldsymbol{v}_h-cdL\boldsymbol{\varphi}_h,$$

$$K\frac{d\boldsymbol{v}_h}{dt}=-Q'\boldsymbol{u}_h,\ M\frac{d\boldsymbol{f}_h}{dt}=cQ\boldsymbol{\varphi}_h-c^2dL(\boldsymbol{\psi}_h-\boldsymbol{v}_h),$$
$$K\frac{d\boldsymbol{\varphi}_h}{dt}=-Q'\boldsymbol{f}_h,$$
$$K\frac{d\boldsymbol{\psi}_h}{dt}=L'\boldsymbol{f}_h,$$

$0<t\le T$, with the initial conditions

$$\boldsymbol{u}_h(0)=\boldsymbol{w}^{(1)},\quad \boldsymbol{v}_h(0)=\boldsymbol{w}^{(2)},\quad \boldsymbol{f}_h(0)=\boldsymbol{\psi}^{(1)},$$
$$\boldsymbol{\varphi}_h(0)=\boldsymbol{\psi}_x^{(2)},\quad \boldsymbol{\psi}_h(0)=\boldsymbol{\psi}^{(2)}$$

is solved by an implicit difference scheme. Denoting the approximate values of vectors $\boldsymbol{u}_h(t_n)$, $\boldsymbol{v}_h(t_n)$, $\boldsymbol{f}_h(t_n)$, $\boldsymbol{\varphi}_h(t_n)$, $\boldsymbol{\psi}_h(t_n)$ by $\boldsymbol{u}_h^n$, $\boldsymbol{v}_h^n$, $\boldsymbol{f}_h^n$, $\boldsymbol{\varphi}_h^n$, $\boldsymbol{\psi}_h^n$, where $t_n=n\tau$, $n=0,1,\dots,P$, $\tau=T/P$ is the time step. Let us represent the difference scheme in the following form

$$\begin{gathered}
M(\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1})=\frac{\tau}{4}\{2(cd-a)+bh[(\boldsymbol{v}_h^n)'K\boldsymbol{v}_h^n\\
+(\boldsymbol{v}_h^{n-1})'K\boldsymbol{v}_h^{n-1}]\}Q(\boldsymbol{v}_h^n+\boldsymbol{v}_h^{n-1})-\frac{\tau cd}{2}L(\boldsymbol{\varphi}_h^n+\boldsymbol{\varphi}_h^{n-1}),2K(\boldsymbol{v}_h^n-\boldsymbol{v}_h^{n-1})\\
=-\tau Q'(\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1}),M(\boldsymbol{f}_h^n-\boldsymbol{f}_h^{n-1})=\frac{\tau c}{2}Q(\boldsymbol{\varphi}_h^n+\boldsymbol{\varphi}_h^{n-1})\\
-\frac{\tau c^2 d}{2}L(\boldsymbol{\psi}_h^n+\boldsymbol{\psi}_h^{n-1}-\boldsymbol{v}_h^n-\boldsymbol{v}_h^{n-1}),\\
2K(\boldsymbol{\varphi}_h^n-\boldsymbol{\varphi}_h^{n-1})=-\tau Q'(\boldsymbol{f}_h^n+\boldsymbol{f}_h^{n-1}),2K(\boldsymbol{\psi}_h^n-\boldsymbol{\psi}_h^{n-1})=\tau L'(\boldsymbol{f}_h^n+\boldsymbol{f}_h^{n-1}),
\end{gathered}\tag{3}$$

$n=1,2,\dots,P$, with the initial conditions

$$\boldsymbol{u}_h^0=\boldsymbol{w}^{(1)},\quad \boldsymbol{v}_h^0=\boldsymbol{w}^{(2)},\quad \boldsymbol{f}_h^0=\boldsymbol{\psi}^{(1)},\quad \boldsymbol{\varphi}_h^0=\boldsymbol{\psi}_x^{(2)},\quad \boldsymbol{\psi}_h^0=\boldsymbol{\psi}^{(2)}.\tag{4}$$

Here $\boldsymbol{w}^{(1)},\boldsymbol{\psi}^{(1)}\in R^{N-1}$, $\boldsymbol{w}^{(2)},\boldsymbol{\psi}^{(2)},\boldsymbol{\psi}_x^{(2)}\in R^{N+1}$ are the known vectors and K, L, M and Q are $(N+1)\times(N+1)$, $(N-1)\times(N+1)$, $(N-1)\times(N-1)$ and $(N-1)\times(N+1)$ matrices of the form

$$K=\frac{1}{6}\begin{pmatrix}2&1&&&&\\1&4&&&&0\\&&\cdot&&&\\&&&\cdot&&\\&&&&\cdot&\\0&&&&4&1\\&&&&1&2\end{pmatrix},\quad L=\frac{1}{6}\begin{pmatrix}1&4&1&&&\\&&\cdot&&&0\\&&&\cdot&&\\0&&&&\cdot&\\&&&1&4&1\end{pmatrix},$$

$$M=\frac{1}{6}\begin{pmatrix}4&1&&&\\1&\cdot&&&0\\&&\cdot&&\\0&&&\cdot&1\\&&&1&4\end{pmatrix},\quad Q=\frac{1}{2h}\begin{pmatrix}-1&0&1&&&\\&&\cdot&&&0\\&&&\cdot&&\\0&&&&\cdot&\\&&&-1&0&1\end{pmatrix}.$$

3 Accuracy of difference scheme. We need some notation. Denote by $\boldsymbol{\alpha}_h^{n,n-1} = (\boldsymbol{\alpha}_{h1}^{n,n-1}, \boldsymbol{\alpha}_{h2}^{n,n-1}, \boldsymbol{\alpha}_{h3}^{n,n-1}, \boldsymbol{\alpha}_{h4}^{n,n-1}, \boldsymbol{\alpha}_{h5}^{n,n-1})'$ the truncation error of scheme (3), (4) where the components $\boldsymbol{\alpha}_{hi}^{n,n-1}$, $i = 1, 2, \dots, 5$ are obtained from (3), (4) as a result of substitution of $\boldsymbol{u}_h^n, \boldsymbol{v}_h^n, \boldsymbol{f}_h^n, \boldsymbol{\varphi}_h^n$, $\boldsymbol{\psi}_h^n$ in $\boldsymbol{u}_h(t_n)$, $\boldsymbol{v}_h(t_n)$, $\boldsymbol{f}_h(t_n)$, $\boldsymbol{\varphi}_h(t_n)$,$\boldsymbol{\psi}_h(t_n)$. Here the symbol $'$ means the operation of transpose. Suppose,

$$\gamma = \max{}^{\frac{1}{2}}(\gamma_1, \gamma_2, \gamma_3)\,,$$

$$\gamma_1 = 2c^4 d^2 + \frac{(cd-a)^2}{h^2} + \frac{5cd}{6h}(c^2 + cd - a), \quad \gamma_2 = 4(1 + \frac{1}{h^2})\,,$$

$$\gamma_3 = c^2 d^2 + \frac{c^2}{h^2} + \frac{5cd}{6h}(2c^2 + cd - a)\,.$$

Let us introduce a vector norm. If we assume that λ and μ are vectors of the same dimension, whose l-th components are equal to λ_l and μ_l, the scalar product

$$(\lambda, \mu)_h = h \sum_l \lambda_l \mu_l\,,$$

where the summation involves all components of λ and μ and the norm

$$\|\lambda\|_h = (\lambda, \lambda)_h^{\frac{1}{2}}.$$

If W is a symmetric positively definite matrix whose order coincides with the dimension of the vector λ, then the energetic norm

$$\|\lambda\|_{W,h} = (W\lambda, \lambda)_h^{\frac{1}{2}}.$$

Assume,

$$s_0 = \frac{a}{b} + \left\{ \frac{2}{b} \left[\|\boldsymbol{w}^{(1)}\|_{M,h}^2 + cd\|\boldsymbol{w}^{(2)} - \boldsymbol{\psi}^{(2)}\|_{K,h}^2 + \frac{1}{2b}(a - b\|\boldsymbol{w}^{(2)}\|_{K,h}^2)^2 \right.\right.$$
$$\left.\left. + \frac{1}{c}\|\boldsymbol{\psi}^{(1)}\|_{M,h}^2 + \|\boldsymbol{\psi}_x^{(2)}\|_{K,h}^2 \right] \right\}^{\frac{1}{2}} + \sqrt{\frac{6}{b}} cdT \|L\boldsymbol{\psi}_x^{(2)} - Q\boldsymbol{\psi}^{(1)}\|_h\,.$$

We formulate the main result on the accuracy of difference scheme (3), (4). By the error of scheme (3), (4) we understood the difference $\boldsymbol{z}_h^n = \boldsymbol{y}_h^n - \boldsymbol{y}_h(t_n)$ between the vectors $\boldsymbol{y}_h^n = (\boldsymbol{u}_h^n, \boldsymbol{v}_h^n, \boldsymbol{f}_h^n, \boldsymbol{\varphi}_h^n, \boldsymbol{\psi}_h^n)'$ and $\boldsymbol{y}_h(t_n) = (\boldsymbol{u}_h(t_n), \boldsymbol{v}_h(t_n), \boldsymbol{f}_h(t_n), \boldsymbol{\varphi}_h(t_n), \boldsymbol{\psi}_h(t_n))'$, $n = 0, 1, \dots, P$. Applying a priori estimates method the following result is proven.

Theorem. *If*

$$\tau \le \frac{1 - \varepsilon}{\sigma},$$

then for the error of scheme (3), (4) *the estimate*

$$\|\boldsymbol{z}_h^n\|_h \le \tau \frac{3}{\varepsilon} \exp\left(2\frac{\sigma}{\varepsilon} t_n\right) \sum_{l=1}^{n} \|\boldsymbol{\alpha}_h^{l,l-1}\|_h$$

holds, where $n = 1, 2, \dots, P$, $0 < \varepsilon < 1$, $\sigma = \frac{3}{2}\Big[\gamma + \frac{b}{h} s_0 \Big(1 + 2\sqrt{6}\Big)\Big]$.

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Received 29.05.2019; revised 16.11.2019; accepted 20.12.2019.

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