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ON A NUMERICAL REALIZATION FOR A TIMOSHENKO TYPE NONLINEAR BEAM EQUATION *

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Abstract. An initial-boundary value problem is considered for the Timoshenko type nonlinear integro-differential equation. In particular, considered is an initial-boundary value problem for the J.Ball integro-differential equation, which describes the dynamic state of a beam. The solution is approximated using the Galerkin method, stabile symmetrical difference scheme and Jacobi iteration method. The algorithm has been approved on tests. The results of recounts are represented in tables.

Keywords and phrases: Timoshenko type nonlinear dynamic beam equation, J.Ball equation, Galerkin method, implicit symmetric difference scheme, Jacobi iterative method, numerical realization.

AMS subject classification (2010): 65M60, 65M06, 65Q10, 65M15.

1 Statement of the problem.

Let us consider the nonlinear equation

$$
\begin{gathered}
u_{tt}(x,t)+\delta u_t(x,t)+\gamma u_{xxxxt}(x,t)+\alpha u_{xxxx}(x,t) \\
-\left(\beta+\rho\int_0^L u_x^2(x,t)\,dx\right)u_{xx}(x,t)-\sigma\left(\int_0^L u_x(x,t)\,u_{xt}(x,t)\,dx\right)\times \\
u_{xx}(x,t)=f(x,t), \quad 0<x<L, \quad 0<t\le T,
\end{gathered}
\tag{1}
$$

with the initial boundary conditions

$$
\begin{gathered}
u(x,0)=u^0(x), \quad u_t(x,0)=u^1(x), \\
u(0,t)=u(L,t)=0, \quad u_{xx}(0,t)=u_{xx}(L,t)=0.
\end{gathered}
\tag{2}
$$

Here $\alpha,\gamma,\rho,\sigma,\beta$ and δ are given constants, among which the first four are positive numbers, while $u^0(x)\in W_2^2(0,L)$ and $u^1(x)\in L_2(0,L)$ are given functions such that $u^0(0)=u^1(0)=u^0(L)=u^1(L)=0$. It will be assumed that the inequality $|\delta|<\gamma\left(\frac{\pi}{L}\right)^4$ is fulfilled when $\delta<0$ and $\alpha\left(\frac{\pi}{L}\right)^2>|\beta|$ holds when $\beta<0$. The equation (1) obtained by J. Ball [1] using the Timoshenko theory, describes the vibration of a beam. The right-hand side $f(x,t)\in L_2((0,L)\times(0,T))$. We suppose that there exits a solution $u(x,t)\in W_2^2((0,L)\times(0,T))$ of problem (1), (2).

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2 Algorithm

2.1 Galerkin method. We write an approximate solution of problem (1), (2) in the form $u_n(x,t)=\sum_{i=1}^{n}u_{ni}(t)\sin\frac{i\pi x}{L}$, where the coefficients $u_{ni}(t)$ will be found by the Galerkin method from the system of ordinary differential equations (see, [2]-[4])

$$
\begin{gathered}
u''_{ni}(t)+\left(\delta+\gamma\left(\frac{i\pi}{L}\right)^4\right)u'_{ni}(t)+\left\{\alpha\left(\frac{i\pi}{L}\right)^4+\left(\frac{i\pi}{L}\right)^2\times\right.\\
\left.\left[\beta+\rho\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2u_{nj}^2(t)+\sigma\frac{L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2u_{nj}(t)\,u'_{nj}(t)\right]\right\}\times\\
u_{ni}(t)=f_{ni}(t),\qquad i=1,2,\cdots,n,\qquad 0<t\le T,
\end{gathered}\tag{3}
$$

with the initial conditions

$$
u_{ni}(0)=a_i^0,\quad u'_{ni}(0)=a_i^1,\quad i=1,2,\cdots,n,\tag{4}
$$

where

$$
a_i^p=\frac{2}{L}\int_0^L u^p(x)\sin\frac{i\pi}{L}x\,dx,\qquad p=0,1,
$$

$$
f_{ni}(t)=\frac{2}{L}\int_0^L f(x,t)\sin\frac{i\pi}{L}x\,dx,\qquad i=1,2,\cdots,n.
$$

2.2 Difference scheme. To solve problem (3), (4) we apply the difference method. On the time interval [0,T] we introduce a net with step $\tau=\frac{T}{M}$ and nodes $t_m=m\,\tau,\ m=0,1,2,\cdots,M$.

On the m-th layer , i.e. for $t=t_m$, the approximate value of $u_{ni}(t_m)$ is denoted by u_{ni}^m, $f_{ni}(t_m)$ by f_{ni}^m. We use an implicit symmetric difference scheme (see, [2]-[4]).

2.3 Iterative method. The system obtained by the discretization will be solved layer-by-layer. Assuming that the solution has already been obtained on the $(m-1)$-th and m-th layer to find it on the $(m+1)$-th layer we use the Jacobi iterative method (see, [2]-[4]).

3 The numerical realization. For approximate solving initial-boundary value problem (1), (2) the several programs in "Maple" is composed and many numerical experiments are carried out.

The algorithm has been approved tests and the results of recounts are represented in tables and graphics (see, [3]).

In this paper we consider the test problems for the following values of geometric (L,T,n,M,H,τ,h) and physical $(\alpha,\gamma,\rho,\sigma,\beta,$ and $\delta)$ parameters:

The length of the beam $L = 1$, the value of the time interval $T = 1$, the number of members in the case of using the Galerkin method $n = 5, 10, 20$, the number of divisions of the time interval $M = 20$, the time step $\tau = \frac{T}{M} = 0.05$, the number of divisions relative to the spatial variable $H = 20$, the step of the spatial variable $h = \frac{L}{H} = 0.05$, the number of iterations $k_0 = 10$, the physical parameters included in the J.Ball equation $\alpha = 1, \gamma = 1, \rho = 1, \sigma = 1, \beta = -1, \delta = -1$.

First of all, we considered test examples that give the error of only the difference and iterative methods

Test 1.1. Exact solution $u(x,t) = \sin \dfrac{\pi x}{L}$,

Test 1.2. Exact solution $u(x,t) = t \sin \dfrac{\pi x}{L}$,

Test 1.3. Exact solution $u(x,t) = t^2 \sin \dfrac{\pi x}{L}$,

Test 1.4. Exact solution $u(x,t) = t^3 \sin \dfrac{\pi x}{L}$,

and then we consider a test example in which, along with the error of the difference and iterative methods, the error of the Galerkin method is taken into account

Test 2.1. Exact solution $u(x,t) = (x^4 - 2x^3 + x)(1 + 3t + t^2))$.

The absolute error for each test problem is calculated by the following formula:

$$error(testN) = \max_{0 \le m,l \le 20} abs\{u_{ex.sol.}(m,l) - u_{appr.sol}(m,l)\},$$

where N number of test.

To find an approximate solution on the first layer, we use difference schemes of $O(\tau)$ or $O(\tau^2)$ order.

$TestN\backslash error$	**n=5**		**n=10**		**n=20**	
	$O(\tau)$	$O(\tau^2)$	$O(\tau)$	$O(\tau^2)$	$O(\tau)$	$O(\tau^2)$
1.1	0	0	0	0	0	0
1.2	0.00007	0.00007	0.00006	0.00006	0.00008	0.00008
1.3	0.0025	0.0011	0.0025	0.0009	0.0025	0.001
1.4	0.0034	0.0034	0.0031	0.0031	0.0033	0.0033
2.1	0.00628	0.00621	0.00634	0.00627	0.00635	0.00628

Conclusion

A) In the case of test examples of the first type, the absolute errors do not change with increasing n, since the basis functions are sinuses. B) In the case of test examples of the second type, an increase in n has a little effect on the score. The increase in n in some cases even spoiled the results of the account.

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