Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 33, 2019

NONPERTURBATIVE EXTENSION OF PERTURBATIVE QUANTUM CHROMODYNAMICS

Nugzar Makhaldiani

Abstract. Introduction in dimensional regularization and renormalization of quantum field theories. Exact infrared (ultraviolet) extension of quantum chromo (electro) dynamics is given.

Keywords and phrases: Renormdynamics, QCD, QED.

AMS subject classification (2010): 11A41, 81T17, 40A05, 81T13, 70H09, 81T15, 81T16.

It is 65 years since Yang and Mills (1954) [1] performed their pioneering work on gauge theories. In the standard model of particle physics, the strong force is described by the theory of quantum chromodynamics (QCD). At ordinary temperatures or densities this force just confines the quarks into composite particles (hadrons) of size around 10^{-15} m = 1 femtometer = 1 fm (corresponding to the QCD energy scale Λ_{QCD}=200 MeV) and its effects are not noticeable at longer distances. However, when the temperature reaches the QCD energy scale (of order 10^{12} kelvins) or the density rises to the point where the average inter-quark separation is less than 1 fm (quark chemical potential around 400 MeV), the hadrons are melted into their constituent quarks, and the strong interaction becomes the dominant feature of the physics. Such phases are called quark matter or QCD matter or Gluquar. The strength of the color force makes the properties of quark matter unlike gas or plasma, instead leading to a state of matter more reminiscent of a liquid. At high densities, quark matter is a Fermi liquid, but is predicted to exhibit color superconductivity at high densities and temperatures below 10^{12} K. Confining property of the background state, asymptotic freedom (AF) and infrared (IR) freezing, effective mass entering the IR freezing logarithms are in good agreement with phenomenology and lattice data. QCD is the theory of the strong interactions with, as only inputs, one mass parameter for each quark species and the value of the QCD coupling constant at some energy or momentum scale in some renormalization scheme. This last free parameter of the theory can be fixed by Λ_{QCD}, the energy scale used as the typical boundary condition for the integration of the Renormdynamic (RD) equation for the strong coupling constant. This is the parameter which expresses the scale of strong interactions, the only parameter in the limit of massless quarks. While the evolution of the coupling with the momentum scale is determined by the quantum corrections induced by the renormalization of the bare coupling and can be computed in perturbation theory, the strength itself of the interaction, given at any scale by the value of the renormalized coupling at this scale, or equivalently by Λ_{QCD}, is one of the above mentioned parameters of the theory and has to be taken from experiment. The RD equations play an important role in our understanding of Quantum Chromodynamics and the strong interactions. The beta function and the quarks mass anomalous dimension are among the most prominent objects for QCD RD equations. The calculation of the one-loop β-function in QCD has lead to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions (see [2]). The MS-scheme belongs to the class of massless schemes where the β-function does not depend on masses of the theory and the first two coefficients of the β-function are scheme-independent.

The Lagrangian of QCD with massive quarks in the covariant gauge is

$$
\begin{aligned}
&L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{q}_n(i\gamma D - m_n)q_n - \frac{1}{2\xi}(\partial A)^2 + \partial^\mu \bar{c}^a(\partial_\mu c^a + gf^{abc}A^b_\mu c^c),\\
&F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu.\ (D_\mu)_{kl} = \delta_{kl}\partial_\mu - igt^a_{kl}A^a_\mu.
\end{aligned}
$$

The notion of asymptotic freedom is basic in establishing QCD as a selfconsistent theory. The extrapolation of the QCD coupling constant as $\alpha_s(Q)$ to larger distances (smaller momenta Q) leads however to inconsistencies of several kinds in the pure (nonbackground) perturbation theory.

The RD equation for the coupling constant is

$$
\begin{aligned}
&\dot{a} = \beta(a) = \beta_2 a^2 + \beta_3 a^3 + \beta_4 a^4 + \beta_5 a^5 + O(a^6),\\
&a = \frac{\alpha_s}{4\pi} = \Big(\frac{g}{4\pi}\Big)^2,\ \int_{a_0}^{a} \frac{da}{\beta(a)} = t - t_0 = \ln\frac{\mu^2}{\mu_0^2},
\end{aligned}
$$

μ is the 't Hooft unit of mass, the renormalization point in the MS-scheme. To calculate the β-function we need to calculate the renormalization constant Z of the coupling constant, $a_b = Za$, where a_b is the bare (unrenormalized) charge. The expression of the β-function can be obtained in the following way

$$
\begin{aligned}
&0 = d(a_b\mu^{2\varepsilon})/dt = \mu^{2\varepsilon}\Big(\varepsilon Za + \frac{\partial(Za)}{\partial a}\frac{da}{dt}\Big)\\
&\Rightarrow \frac{da}{dt} = \beta(a,\varepsilon) = \frac{-\varepsilon Za}{\frac{\partial(Za)}{\partial a}} = -\varepsilon a + \beta(a),\ \beta(a) = a\frac{d}{da}(aZ_1),
\end{aligned}
$$

where

$$
\beta(a,\varepsilon) = \frac{D-4}{2}a + \beta(a)
$$

is $D-$dimensional $\beta-$function and Z_1 is the residue of the first pole in ε expansion

$$
Z(a,\varepsilon) = 1 + Z_1\varepsilon^{-1} + \cdots + Z_n\varepsilon^{-n} + \cdots . \tag{1}
$$

Since Z does not depend explicitly on μ, the β-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ. Note that, presentation of Z in the form of expansion (1) is formal. If we take $\varepsilon = 1/p$, we can give the expansion p-adic sense [3]. So, we will have renormalization factors Z as analytic functions of p-adic argument. RD equation,

$$
\dot{a} = \beta_1 a + \beta_2 a^2 + \cdots
$$

can be reparametrized,

$$
\begin{aligned}
&a(t) = f(A(t)) = A + f_2A^2 + ... + f_nA^n + ... = \sum_{n\geq 1} f_nA^n,\\
&\dot{A} = b_1A + b_2A^2 + ... = \sum_{n\geq 1} b_nA^n,
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&\dot{a} = \dot{A}f'(A) = (b_1A + b_2A^2 + ...)(1 + 2f_2A + ... + nf_nA^{n-1} + ...)\\
&= \beta_1(A + f_2A^2 + ... + f_nA^n + ...) + \beta_2(A^2 + 2f_2A^3 + ...) + ...\\
&+\beta_n(A^n + nf_2A^{n+1} + ...) + ... = \beta_1A + (\beta_2 + \beta_1f_2)A^2\\
&+(\beta_3 + 2\beta_2f_2 + \beta_1f_3)A^3 + ... + (\beta_n + (n-1)\beta_{n-1}f_2 + ... + \beta_1f_n)A^n + ...\\
&= \sum_{n,n_1,n_2\geq 1} A^n b_{n_1} n_2 f_{n_2} \delta_{n,n_1+n_2-1}\\
&= \sum_{n,m\geq 1;m_1,...,m_k\geq 0} A^n\beta_m f_1^{m_1}...f_k^{m_k} f(n,m,m_1,...,m_k),
\end{aligned}
$$

$$
\begin{aligned}
&f(n,m,m_1,...,m_k) = \frac{m!}{m_1!...m_k!}\delta_{n,m_1+2m_2+...+km_k}\delta_{m,m_1+m_2+...+m_k},\\
&b_1 = \beta_1,\ b_2 = \beta_2 + f_2\beta_1 - 2f_2b_1 = \beta_2 - f_2\beta_1,\\
&b_3 = \beta_3 + 2f_2\beta_2 + f_3\beta_1 - 2f_2b_2 - 3f_3b_1 = \beta_3 + 2(f_2^2 - f_3)\beta_1,\\
&b_4 = \beta_4 + 3f_2\beta_3 + f_2^2\beta_2 + 2f_3\beta_2 - 3f_4b_1 - 3f_3b_2 - 2f_2b_3, ...,\\
&b_n = \beta_n + ... + \beta_1f_n - 2f_2b_{n-1} - ... - nf_nb_1, ...
\end{aligned}
\tag{3}
$$

So, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but β_1. We can fix any higher coefficient with zero value, if we take

$$
f_2 = \frac{\beta_2}{\beta_1},\ f_3 = \frac{\beta_3}{2\beta_1} + f_2^2,\ ...\ , f_n = \frac{\beta_n + ...}{(n-1)\beta_1}, ...
$$

In the critical dimension of space-time, $\beta_1 = 0$, and we can change by reparametrization any coefficient but β_2 and β_3. From the relations (3), in the critical dimension ($\beta_1 = 0$), we find that, we can define the minimal form of the RD equation

$$
\dot{A} = \beta_2A^2 + \beta_3A^3. \tag{4}
$$

We can solve (4) as implicit function,

$$
u^{\beta_3/\beta_2}e^{-u} = ce^{\beta_2 t},\ u = \frac{1}{A} + \frac{\beta_3}{\beta_2},
$$

then, as in the noncritical case, explicit solution will be given by reparametrization representation (2) [4]. If we know somehow the coefficients β_n, e.g. for first several exact and for others asymptotic values (see e.g. [5]) than we can construct reparametrization function (2) and find the dynamics of the running coupling constant. This is similar to the action-angular canonical transformation of the analytic mechanics. Statement:

The reparametrization series for a is p-adically convergent, when β_n and A are rational numbers. Let us solve fundamental equation of RD (4)

$$\begin{aligned}&\frac{dA}{\beta_2 A^3(1/A+\beta_3/\beta_2)} = dt \Rightarrow \frac{d(1/A)1/A}{1/A+\beta_3/\beta_2} = -\beta_2 dt \Downarrow\\ &x - a\ln(x+a) = -\beta_2 t + c,\ x = 1/A,\ a = \beta_3/\beta_2.\end{aligned} \tag{5}$$

Nonperturbative extension means the following change

$$t = \ln\frac{p^2}{\Lambda^2} \to t_m = \ln\frac{p^2+m^2}{\Lambda^2},\ \frac{dt_m}{dt} = \frac{p^2}{p^2+m^2},$$

in the solution (5). Let us fined corresponding RD motion equation

$$\dot{x}\left(1-\frac{a}{x+a}\right) = -\beta_2\frac{p^2}{p^2+m^2} \Downarrow\ \ \dot{A} = (\beta_2 A^2+\beta_3 A^3)\frac{p^2}{p^2+m^2} = \left\{\begin{array}{cc}\beta_{pert}, & p^2 \gg m^2,\\ 0, & p^2 \ll m^2,\end{array}\right.$$

$$\frac{p^2}{p^2+m^2} = 1-\frac{m^2}{\Lambda^2}e^{(1/A-c)/\beta_2}(1/A+\beta_3/\beta_2)^{-\beta_3/\beta_2^2}.$$

If we have infrared asymptotic freedom (as in QED) and ultraviolet fixed point, we take the modified time variable as

$$t_m = \ln\left(\frac{m^2/\Lambda^2}{1+m^2/p^2}\right),\ \frac{dt_m}{dt} = \frac{m^2/p^2}{1+m^2/p^2} = \frac{m^2}{m^2+p^2} = \left\{\begin{array}{ll}1 & p^2 \ll m^2\\ m^2/p^2 & p^2 \gg m^2.\end{array}\right.$$

The extended renormdynamics motion equation is

$$\dot{A} = (\beta_2 A^2+\beta_3 A^3)\frac{m^2}{p^2+m^2} = \left\{\begin{array}{cc}\beta_{pert}, & p^2 \ll m^2,\\ 0, & p^2 \gg m^2,\end{array}\right.$$

$$\frac{m^2}{p^2+m^2} = 1-\frac{\Lambda^2}{m^2}e^{-(1/A-c)/\beta_2}(1/A+\beta_3/\beta_2)^{\beta_3/\beta_2^2}.$$

REFERENCES

1. Yang, C.N., Mills, R.L. Conservation of isotopic spin and isotopic gauge invariance. *Phys. Rev.*, **96** (1954), 191-195.
2. Collins, J.C. Renormalization. *Cambrige Univ. Press, London*, 1984.
3. Koblitz, N. p-Adic Numbers, p-Adic Analysis, and Zeta-functions. *Springer-Verlag, New York*, 1977.
4. Makhaldiani, N. Renormdynamics, unified field theories and universal distributions of the multipartical production processes. *Physics of Particles and Nuclei Letters*, **11** (2014), 10-19.
5. Kazakov, D.I., Shirkov, D.V. Asymptotic series in quantum field theory and their summation. *Fortschr. der Phys.*, **28** (1980), 465-499.

Received 30.05.2019; revised 28.11.2019; accepted 26.12.2019.

Author(s) address(es):

Nugzar Makhaldiani
Joint Institute for Nuclear Research
Joliot-Curie str. 6, 141980 Dubna, Moscow region, Russia
E-mail: mnv@jinr.ru