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A REPREZENTATION FORMULA OF A GENERAL SOLUTION OF THE HOMOGENEOUS SYSTEM OF DIFFERENTIAL EQUATIONS FOR THE MICROSTRETCH MATERIALS WITH MICROSTRUCTURE AND MICROTEMPERATURES

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Abstract. We consider the stationary oscillations of the Microstretch materials with microstructure and microtemperatures. The representation formula of a general solution of the homogeneous system of differential equations obtained in the paper is expressed by means of seven metaharmonic functions. These formulas are very convenient in many particular problems for domains with concrete geometry.

Keywords and phrases: Microstretch materials, metaharmonic, microtemperatures.

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1 Introduction. The effective solution of various boundary value and contact problems of the elasticity theory (of classical and generalized models) is very important from theoretical and practical standpoints. Unfortunately, exact solutions of these problems can be constructed explicity only for a rather limited number of bodies of concrete geometrical form. Here an essential role is played by the fact that a general solution of the system of complex differential equations that corespond to a mechanical model can be represented by solutions of simpler differential equations (of Helmholtz). The present paper deals with the representation of solutions of a system of differential equations of stationary oscillation for thermoelasticity of microstretch materials with microtemperatures,with the construction of explicit solutions of the Dirichlet and Neumann problems for a circle and hollow circle, and with mathematical investigation of these solutions.

2 Basic equations and fundamental theorem. The system of homogeneous differential equations of the stationary oscillation of the thermoelasticity theory of microstretch materials with microtemperatures and microdilatation in the case of isotropic bodies is written in the form [1]

$$
\begin{gathered}
\big((\mu+\varkappa)\Delta+\rho\sigma^2\big)u+(\lambda+\mu)\operatorname{grad}\operatorname{div}u-\varkappa\operatorname{rot}\omega+\mu_0\operatorname{grad}v-\beta_0\operatorname{grad}\theta=0,\\
(\varkappa_6\Delta+\varkappa_0)w+(\varkappa_4+\varkappa_5)\operatorname{grad}\operatorname{div}w+i\sigma\mu_1\operatorname{rot}\omega+i\sigma\mu_2\operatorname{grad}v-\varkappa_3\operatorname{grad}\theta=0,\\
(\gamma\Delta+\delta)\omega+\varkappa\operatorname{rot}u-\mu_1\operatorname{rot}w=0, \qquad (1)\\
(a_0\Delta+\eta_0)v-\mu_0\operatorname{div}u-\mu_2\operatorname{div}w+\beta_1\theta=0,\\
(\varkappa_7\Delta+i\sigma c)\theta+i\beta_0T_0\sigma\operatorname{div}u+\varkappa_1\operatorname{div}w+i\beta_1T_0\sigma v=0,
\end{gathered}
$$

where Δ is the two-dimensional Laplace operator, $u=(u_1,u_2)^\top$ is a displacement vector, $w=(w_1,w_2)^\top$ is a microtemperature vector, ω is the microrotation function, v is the microdilatation function, θ is the temperature, measured from a fixed absolute temperature T_0 ($T_0>0$), σ is a frequency parameter, $\sigma=\sigma_1+i\sigma_2$, $\sigma_2>0$, $\sigma_1\in R$, $\delta=I_1\sigma^2-2\varkappa$, $\varkappa_0=i\sigma b-\varkappa_2$, $\eta_0=I\sigma^2-\eta$, $c=aT_0$, λ, μ, $\varkappa$, η, β_0, β_1, μ_0, μ_1, μ_2, a_0, b, I, I_1, are the real constants characterizing the mechanical and thermal properties of the body, ρ is the mass density, $\top$ is the transposition symbol, here

$$\operatorname{rot}:=\Big(-\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1}\Big)^\top,\qquad \operatorname{rot}\omega:=\Big(-\frac{\partial\omega}{\partial x_2},\frac{\partial\omega}{\partial x_1}\Big)^\top,$$
$$\operatorname{rot}u:=\operatorname{rot}\cdot u=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2},\qquad \operatorname{rot}w:=\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}.$$

The following theorem is valid.

Theorem 1. *A vector $U=(u,w,\omega,v,\theta)^\top\in C^2(\Omega)$ is a solution of system* (1) *in a domain $\Omega\in R^2$ if and only if is representable in the form*

$$u(x)=\sum_{j=1}^7 u^{(i)}(x),\quad w(x)=\sum_{j=1}^7\alpha_j u^{(j)}(x),\quad \omega(x)=\sum_{j=5}^7\gamma_j\operatorname{rot}u^{(j)}(x),$$
$$v(x)=\sum_{j=1}^4\gamma_j\operatorname{div}u^{(j)}(x),\quad \theta(x)=\sum_{j=1}^4\delta_j\operatorname{div}u^{(j)}(x),$$

where

$$\big(\Delta+k_j^2\big)u^{(j)}(x)=0,\quad \operatorname{div}u^{(j)}(x)=0,\quad j=5,6,7,$$
$$\big(\Delta+k_j^2\big)u^{(j)}(x)=0,\quad \operatorname{rot}u^{(j)}(x)=0,\quad j=1,2,3,4,$$
$$\alpha_j=\frac{1}{a_1}\big[(\lambda_0k_j^2-\rho\sigma^2)(a_0\varkappa_3k_j^2-\eta_0\varkappa_3-i\sigma\beta_1\mu_2)-\mu_0(\mu_0\varkappa_3-i\sigma\beta_0\mu_2)k_j^2\big],$$
$$\gamma_j=\frac{1}{a_1k_j^2}\big[(\lambda_0k_j^2-\rho\sigma^2)\big(\beta_1(l_0k_j^2-\varkappa_0)-\mu_2\varkappa_3k_j^2\big)-\beta_0\mu_0(l_0k_j^2-\varkappa_0)k_j^2\big],$$
$$\delta_j=\frac{1}{a_1k_j^2}\big[(\lambda_0k_j^2-\rho\sigma^2)((l_0k_j^2-\varkappa_0)(a_0k_j^2-\eta_0)-i\sigma\mu_2^2k_j^2)$$
$$-\mu_0^2(l_0k_j^2-\varkappa_0)k_j^2\big],\quad j=1,2,3,4,$$

$$\alpha_j=\frac{i\sigma\mu_1((\mu+\varkappa)k_j^2-\rho\sigma^2)}{\varkappa(\varkappa_0-\varkappa_6k_j^2)},\quad \gamma_j=\frac{(\mu+\varkappa)k_j^2-\rho\sigma^2}{\varkappa k_j^2},\quad j=5,6,7,$$
$$a_1=a_0l_0\beta_0k_j^4-\big[\beta_0(a_0\varkappa_0+l_0\eta_0+i\sigma\mu_2^2)+\mu_0(l_0\beta_1-\mu_2\varkappa_3)\big]k_j^2+\varkappa_0(\mu_0\beta_1+\beta_0\eta_0),$$
$$\lambda_0=\lambda+2\mu+\varkappa,\quad l_0=\varkappa_4+\varkappa_5+\varkappa_6.$$

Here $-k_j^2$, $j=1,2,3,4$ and k_j^2, $j=5,6,7$, are the roots of the equations $\Lambda_1(\Delta)=0$, and $\Lambda_2(\mathrm{rot})=0$ respectively, where

$$\Lambda_1(\Delta)=\frac{1}{d_0}\det\begin{bmatrix}\lambda_0\Delta+\rho\sigma^2 & 0 & \mu_0\Delta & -\beta_0\Delta\\ 0 & l_0\Delta+\varkappa_0 & i\sigma\mu_2\Delta & -\varkappa_3\Delta\\ -\mu_0 & -\mu_2 & a_0\Delta+\eta_0 & \beta_1\\ i\sigma\beta_0T_0 & \varkappa_1 & i\sigma\beta_1T_0 & \varkappa_7\Delta+i\sigma c\end{bmatrix},$$

$$\Lambda_2(\mathrm{rot})=\frac{1}{p_0}\begin{bmatrix}\rho\sigma^2+(\mu+\varkappa)\,\mathrm{rot}\,\mathrm{rot} & 0 & -\varkappa\,\mathrm{rot}\,\mathrm{rot}\\ 0 & \varkappa_0+\varkappa_6\,\mathrm{rot}\,\mathrm{rot} & i\sigma\mu_1\,\mathrm{rot}\,\mathrm{rot}\\ \varkappa & -\mu_1 & \sigma+\gamma\,\mathrm{rot}\,\mathrm{rot}\end{bmatrix},$$

$$d_0=a_0\lambda_0l_0\varkappa_7,\quad p_0=\gamma\varkappa_6(\mu+\varkappa).$$

Theorem 2. *A vector $U=(u,w,\omega,v,\theta)^\top\in C^2(\Omega)$ is a general solution of the homogeneous system* (1) *in a domain $\Omega\in R^2$ if and only if it is representable in the form*

$$\begin{aligned}
&u(x)=\sum_{j=1}^{4}\mathrm{grad}\,\phi_j(x)+\sum_{j=5}^{7}\mathrm{rot}\,\phi_j(x),\\
&w(x)=\sum_{j=1}^{4}\alpha_j\,\mathrm{grad}\,\phi_j(x)+\sum_{j=5}^{7}\alpha_j\,\mathrm{rot}\,\phi_j(x),\\
&\omega(x)=-\sum_{j=5}^{7}\gamma_jk_j^2\phi_j(x),\quad v(x)=-\sum_{j=1}^{5}\gamma_jk_j^2\phi_j(x),\\
&\theta=-\sum_{j=1}^{4}\delta_jk_j^2\phi_j(x),\\
&(\Delta+k_j^2)\phi_j=0,\quad j=1,2,\dots,7.
\end{aligned}\tag{2}$$

3 Solution of the boundary value problems. Let $\Omega^+\subset R^2$ be a circle, bounded by the circumference $\partial\Omega$, centered at the origin and having radius R. We denote $\Omega^-:=R^2\backslash\overline{\Omega}^+$.

Problem. Find, in the domain Ω^+ (Ω^-), a regular vector $U=(u,w,\omega,v,\theta)^\top\in C^2(\Omega^\pm)\cap C^1(\overline{\Omega}^\pm)$ such that it satisfies in this domain the system of differential equations (1) and, on the boundary $\partial\Omega$, satisfies one of the following boundary conditions:

$\overset{\sigma}{(\mathrm{I})}{}^\pm$ **(Dirichlet problem)**

$$\{U(z)\}^\pm=f(z),\qquad z\in\partial\Omega,$$

$\overset{\sigma}{(\mathrm{II})}{}^\pm$ **(Neumann problem)**

$$\{P(\partial,n)U(z)\}^\pm=f(z),\qquad z\in\partial\Omega,$$

where $f = (f^{(1)}, f^{(2)}, f_3, f_4, f_5)^\top$, $f^{(i)} = (f_1^{(j)}, f_2^{j})^\top$, $j = 1,2$, $f_k^{(j)}$, $k,j = 1,2$, f_l, $l = 3,4,5$ are the functions, given on the boundary $\partial\Omega$, $n(z)$ is the outward normal unit vector passing at a poind $z \in \partial\Omega$ with respect to the domain Ω^+, and $P(\partial, n)U$ is the generalized thermo-stress vector. In the case of the exterior problems for the domain Ω^- the vector $U = (u, w, \omega, v, \theta)^\top$ should satisfy the following decay conditions at infinity

$$U(x) = O(|x|^{-1}), \quad \partial_k U(x) = o(|x|^{-1}), \quad k = 1,2.$$

Solutions of Dirichlet and Neumann problems will be sought for in form (2), where the functions $\phi_j(x)$, $j = 1,2,\dots,7$, are represented as

$$\phi_j(x) = \sum_{k=0}^{\infty} g_k(k_j r)(a_k^{(j)} \cos k\varphi + b_k^{(j)} \sin k\varphi), \quad j = 1,2,\dots,7,$$

where $a_k^{(j)}$, $b_k^{(j)}$, are the unknown constants,

$$g_k(k_j r) = \begin{cases} \dfrac{J_k(k_j r)}{J_k(k_j R)}, & r < R, \\ \dfrac{H_k^{(1)}(k_j r)}{H_k^{(1)}(k_j R)}, & r > R, \end{cases}$$

$J_k(x)$ is the Bessel function, $H_k^{(1)}(x)$ is the Hankel function.

Solutions of Dirichlet and Neumann problems are obtained in terms of absolutely and uniformly convergent series.

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