Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 33, 2019

LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS APPROACH WITH PIECEWISE CONSTANT CUT-OFF FUNCTION FOR THE DIRICHLET PROBLEM OF THE HEAT TRANSFER EQUATION WITH A VARIABLE COEFFICIENT

Sveta Gorgisheli

Abstract. A localized boundary-domain integro-differential equations system associated with the Dirichlet boundary value problem for the stationary heat transfer partial differential equation with a variable coefficient is obtained and analysed.

Keywords and phrases: Localized parametrix, localized potential method.

AMS subject classification (2010): 31B10.

In the paper the *localized boundary-domain integro-differential equations* (LBDIDE) system associated with the Dirichlet boundary value problem (BVP) for the stationary heat transfer partial differential equation with a variable coefficient is obtained and analysed. The parametrix is localized by a characteristic function of a ball of radius ε which is not a smooth cut-off function in the whole space. For smooth localizing cut-off functions this method is theoretically studied and substantiated in [1], [2], where the BVPs are reduced to systems of *Localized boundary-domain integral equations*.

The main results of the present paper are equivalence theorems of the LBDIDE systems to the original variable-coefficient BVPs and unique solvability of the LBDIDE systems in the corresponding Sobolev spaces.

Let Ω be a bounded region of $\mathbb{R}^3$ surrounded by a simply connected smooth Liapunov surface $S=\partial\Omega\in C^{2,\alpha}$ with $\alpha>0$. Let $B(y,\varepsilon):=\{x\in\mathbb{R}^3\ :\ |x-y|\leqslant\varepsilon\}$ be a ball centered at y and radius ε, where ε is a fixed positive number, and $\Sigma(y,\varepsilon):=\partial B(y,\varepsilon)$.Further, let

$$\begin{aligned} &\Omega(y,\varepsilon):=\Omega\cap B(y,\varepsilon), && S(y,\varepsilon):=S\cap B(y,\varepsilon),\\ &\Sigma_1(y,\varepsilon):=\Sigma(y,\varepsilon)\cap\Omega, && \ell(y,\varepsilon):=\partial\Sigma_1(y,\varepsilon)=\partial S(y,\varepsilon). \end{aligned} \tag{1}$$

It is evident that if the distance from the point y to the boundary $S=\partial\Omega$ is grater than ε, $\operatorname{dist}(y;S)>\varepsilon$, then $S(y,\varepsilon)=\varnothing$ and $\Sigma_1(y,\varepsilon)=\Sigma(y,\varepsilon)$. Note also that for $y\in\overline{\Omega}$ the part of the spherical surface $\Sigma_1(y,\varepsilon)$ always possesses a positive measure.

We assume that for a given domain Ω there is $\varepsilon_0>0$, such that for arbitrary $y\in\overline{\Omega}$ and $0<\varepsilon<\varepsilon_0$ the corresponding domain $\Omega(y,\varepsilon)$ is a piecewise smooth Lipschitz domain. Notice that this condition is satisfied for a convex domain and for a domain with a smooth Lyapunov boundary $S=\partial\Omega\in C^{1,\alpha}$, $\alpha>0$. We need this condition to write the corresponding Green identities in the domain $\Omega(y,\varepsilon)$, $y\in\overline{\Omega}$, and also to establish mapping properties for integral operators involved in our analysis.

Introduce a harmonic localized parametrix

$$P_\chi(x) := -\frac{\chi(x)}{4\,\pi\,|x|}\,, \qquad \chi(x) := \begin{cases} 1 & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| > \varepsilon. \end{cases} \tag{2}$$

For $f \in H^0(\Omega)$ we consider the following scalar elliptic differential equation

$$A(x,\partial_x)u(x) := \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\Big(a(x)\frac{\partial u(x)}{\partial x_k}\Big) = f(x), \quad x \in \Omega, \tag{3}$$

$$a \in C^2(\overline{\Omega}), \quad 0 < a_0 \leqslant a(x) \leqslant a_1, \quad \forall\, x \in \overline{\Omega}. \tag{4}$$

A solution function u is sought in the space $H^{1,0}(\Omega, A) = \{v \in H^1(\Omega) \,:\, Av \in H^0(\Omega)\}$.

For an arbitrary piecewise smooth Lipschitz domain $\Omega_1 \subseteq \Omega$, by $\gamma^+ = \gamma^+_{\partial\Omega_1}$ we denote the trace operator on $\partial\Omega_1$ and $n(x)$ is the unit normal vector at the point $x \in \partial\Omega_1$ directed outward Ω_1.

With the help of Green's first identity for an arbitrary function $u \in H^{1,0}(\Omega_1, A)$ we can define on $\partial\Omega_1$ the *canonical conormal derivative* $T^+u \equiv T^+_{\partial\Omega_1}u = a\,\frac{\partial u}{\partial n} \in H^{-\frac{1}{2}}(\partial\Omega_1)$ by the relation

$$\big\langle T^+u\,,\, g\big\rangle_{\partial\Omega_1} := \int_{\Omega_1} \sum_{k=1}^{3} a\,\frac{\partial u}{\partial x_k}\,\frac{\partial v}{\partial x_k}\,dx + \int_{\Omega_1} v\,Au\,dx, \qquad \forall\, g \in H^{\frac{1}{2}}(\partial\Omega_1), \tag{5}$$

where $v \in H^1(\Omega_1)$ and $\gamma^+_{\partial\Omega_1}v = g$. Below we drop the subscript $\partial\Omega_1$ in the notation of trace operator and conormal derivative operator when it does not lead to misunderstanding.

For arbitrary functions $u, v \in H^{1,0}(\Omega_1, A)$ then we have the following Green first and second identities

$$\int_{\Omega_1} v\,Au\,dx + \int_{\Omega_1} \sum_{k=1}^{3} a\,\frac{\partial u}{\partial x_k}\,\frac{\partial v}{\partial x_k}\,dx = \big\langle T^+u\,,\,\gamma^+v\big\rangle_{\partial\Omega_1}, \tag{6}$$

$$\int_{\Omega_1} \big(v\,Au - u\,Av\big)\,dx = \big\langle T^+u\,,\,\gamma^+v\big\rangle_{\partial\Omega_1} - \big\langle T^+v\,,\,\gamma^+u\big\rangle_{\partial\Omega_1}. \tag{7}$$

Remark. Here and in what follows the angled brackets should be understood as duality pairing of $H^{-\frac{1}{2}}(\partial\Omega_1)$ with $H^{\frac{1}{2}}(\partial\Omega_1)$. In the case of a proper sub-manifold $S_1 \subset \partial\Omega_1$ with piecewise smooth Lipschitz boundary curve $\partial S_1 \neq \varnothing$ (e.g., $S(y,\varepsilon)$ or $\Sigma_1(y,\varepsilon)$ for $\operatorname{dist}(y,S) < \varepsilon$) the angled brackets denote duality pairing of either $H^{-\frac{1}{2}}(S_1)$ with $\widetilde{H}^{\frac{1}{2}}(S_1)$ or $\widetilde{H}^{-\frac{1}{2}}(S_1)$ with $H^{\frac{1}{2}}(S_1)$, where $\widetilde{H}^s(S_1)$ and $H^s(S_1)$ are mutually adjoint spaces defined as follows

$$\widetilde{H}^s(S_1) := \big\{g \in H^s(\partial\Omega_1) \,:\, supp \quad g \subseteq \overline{S}_1\big\}, \qquad H^s(S_1) := \big\{r_{S_1}g \,:\, g \in H^s(\partial\Omega_1)\big\}.$$

Here r_{S_1} is the restriction operator onto S_1.

Using Green's second formula (7) for the domain $\Omega_1 = \Omega(y,\varepsilon) \setminus B(y,\delta)$ with $\delta \in (0,\varepsilon)$ and for the functions $u \in H^{1,0}(\Omega, A)$ and $P_\chi(y - \cdot) \in H^{1,0}(\Omega \setminus B(y,\delta), A)$, and passing to the limit as $\delta \to 0$, by standard arguments one can derive Green's third formula

$$a(y)\, u(y) + \mathcal{R}_\varepsilon\, u(y) - V_\varepsilon(T^+u)(y) + W_\varepsilon(\gamma^+u)(y) = \mathcal{P}_\varepsilon(Au)(y), \quad \forall\, y \in \Omega, \tag{8}$$

where γ^+u and T^+u are respectively the trace of u and the canonical conormal derivative of u on the boundary $\partial\Omega(y,\varepsilon) = S(y,\varepsilon) \cup \Sigma_1(y,\varepsilon) \cup \ell(y,\varepsilon)$,

$$\gamma^+u \in H^{\frac{1}{2}}(\partial\Omega(y,\varepsilon)), \quad T^+u \in H^{-\frac{1}{2}}(\partial\Omega(y,\varepsilon)), \tag{9}$$

$\mathcal{R}_\varepsilon$ is a localized weakly singular integral operator

$$\mathcal{R}_\varepsilon\, u(y) := \lim_{\delta\to 0} \int_{\Omega(y,\varepsilon)\setminus B(y,\delta)} [A(x,\partial)P_\chi(x-y)]u(x)\, dx = \int_{\Omega(y,\varepsilon)} R(x,y)\, u(x)\, dx,$$

$$R(x,y) := -\frac{1}{4\,\pi} \sum_{k=1}^{3} \frac{\partial a(x)}{\partial x_k} \frac{\partial}{\partial x_k} \frac{1}{|x-y|} = \mathcal{O}(|x-y|^{-2}) \quad \text{for} \quad x \in \Omega(y,\varepsilon);$$

V_ε, W_ε, and $\mathcal{P}_\varepsilon$ are the localized single layer, double layer, and Newtonian volume type potentials respectively,

$$V_\varepsilon\,(T^+u)(y) := \frac{1}{4\pi} \int_{S(y,\varepsilon)\cup\Sigma_1(y,\varepsilon)} \frac{1}{|x-y|}\, T^+u(x)\, dS_x, \tag{10}$$

$$W_\varepsilon\,(\gamma^+u)(y) := \frac{1}{4\pi} \int_{S(y,\varepsilon)\cup\Sigma_1(y,\varepsilon)} \left[T(x,\partial)\frac{1}{|x-y|}\right] \gamma^+u(x)\, dS_x, \tag{11}$$

$$\mathcal{P}_\varepsilon\,(Au)(y) := \int_{\Omega(y,\varepsilon)} P_\chi(x-y)\, Au(x)\, dx = -\frac{1}{4\pi} \int\limits_{\Omega(y,\varepsilon)} \frac{1}{|x-y|}\, Au(x)\, dx. \tag{12}$$

The trace on S of Green's third formula (8) exists and reads as

$$\gamma_S^+\mathcal{R}_\varepsilon\, u(y) - \mathcal{V}_\varepsilon(T^+u)(y) + \frac{1}{2}\, a(y)\gamma_S^+u(y) + \mathcal{W}_\varepsilon(\gamma^+u)(y) = \gamma_S^+\mathcal{P}_\varepsilon(Au)(y), \;\; \forall\, y \in S. \tag{13}$$

The following auxiliary lemma plays a crucial role in our analysis.

Lemma. *Let ε be a fixed positive number and let $\Omega(y,\varepsilon)$ be the domain defined in* (1)*. Let $g \in \widetilde{H}^0(\Omega)$ and*

$$\int_{\Omega(y,\varepsilon)} \frac{1}{|x-y|}\, g(x)\, dx = 0, \qquad \forall y \in \Omega. \tag{14}$$

Then $g = 0$ in Ω.

Now let us consider the Dirichlet problem for the operator $A(x,\partial)$ defined in (3): Find a function $u \in H^{1,0}(\Omega, A)$ such that

$$A(x,\partial_x)u = f \;\text{ in }\; \Omega, \quad f \in H^0(\Omega), \tag{15}$$

$$\gamma^+u = \varphi_0 \;\text{ on }\; S, \qquad \varphi_0 \in H^{\frac{1}{2}}(S). \tag{16}$$

It is a well known classical result that the Dirichlet problem (15)–(16) is uniquely solvable (see, e.g., [3]).

Substituting the data of the Dirichlet problem under consideration into Green's third formula (8) and into its trace formula (13) on S, we obtain the following system of localized boundary-domain integro-differential equations with respect to the unknown function u,

$$a\,u + \mathcal{R}_\varepsilon\,u - V_\varepsilon(T^+u) + W_\varepsilon(\gamma^+u) = \mathcal{P}_\varepsilon f \quad \text{in} \quad \Omega, \tag{17}$$

$$\gamma_S^+\mathcal{R}_\varepsilon\,u - \mathcal{V}_\varepsilon(T^+u) + \frac{1}{2}\,a(y)\,\varphi_0 + \mathcal{W}_\varepsilon(\gamma^+u) = \gamma_S^+\mathcal{P}_\varepsilon(f) \quad \text{on} \quad S, \tag{18}$$

where the traces of the densities in the potential type operators are taken on the integration surface $\partial\Omega(y,\varepsilon) = S(y,\varepsilon) \cup \Sigma_1(y,\varepsilon) \cup \ell(y,\varepsilon)$.

There holds the following equivalence theorem.

Theorem 1. *Let $f \in H^0(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. The Dirichlet problem* (15)–(16) *and the system of localized boundary-domain integro-differential equations*(17)–(18) *are equivalent in the following sense:*

(i) If $u \in H^{1,0}(\Omega, A)$ solves the Dirichlet problem (15)–(16)*, then u is a solution to the system of localized boundary-domain integro-differential equations* (17)–(18)*, and vice versa,*

(ii) If $u \in H^{1,0}(\Omega, A)$ solves the system of localized boundary domain integro-differential equations (17)–(18)*, then u is a solution to the Dirichlet problem* (15)–(16)*.*

This theorem implies the following existence result.

Theorem 2. *Given $f \in H^0(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, the system of localized boundary-domain integro-differential equations* (17)–(18) *is uniquely solvable in the space $H^{1,0}(\Omega, A)$.*

REFERENCES

1. Chkadua, O., Mikhailov, S.E., Natroshvili, D. Analysis of some localized boundary-domain integral equations. *J. Integral Equations Appl.*, **21**, 3 (2009), 405-445.
2. Chkadua, O., Mikhailov, S.E., Natroshvili, D. Localized boundary-domain singular integral equations based on harmonic parametrix for divergence-form elliptic PDEs with variable matrix coefficients. *Integral Equations and Operator Theory*, **76**, 4 (2013), 509-547.
3. Dautray, R., Lions, J. it Mathematical Analysis and Numerical Methods for Science and Technology. *volume 4: Integral Equations and Numerical Methods. Springer, Berlin–Heidelberg–New York*, 1990.

Received 27.05.2019; revised 11.11.2019; accepted 15.12.2019.

Author(s) address(es):

Sveta Gorgisheli
Georgian Technical University, Department of Mathematics
M. Kostava str. 77, 0160 Tbilisi, Georgia
E-mail: 18barabare@gmail.com