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THE EULER INTEGRAL OF THE FIRST KIND. THE HIGH ORDER SINGULARITY *

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Abstract. It is shown, that the Euler integral of the first kind (beta integral) in some area of its divergence is integrable in the sense of generalized functions. The equality of the mentioned integral and the Fourier transform of a singular exponential function is shown. The connection between the beta integral and the complex Dirac delta function is obtained. In addition, the analytical representation and the asymptotic behavior of the Euler beta functional are derived.

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1 The high order singularity. We studied the Euler Integral of the first kind (the Euler beta Integral) with imaginary parameters and it is shown that the following limiting behavior takes place [1]:

$$B\left(i\gamma,\,-i\gamma\right)=\lim_{\varepsilon\to 0+}B(\varepsilon+i\gamma,\varepsilon-i\gamma)=2\pi\delta(\gamma). \tag{1}$$

The function defined by the formula (1) has the following explicit form:

$$B\left(i\gamma,\,-i\gamma\right)=\int\limits_0^1 t^{i\gamma-1}(1-t)^{-i\gamma-1}dt.$$

Now let's consider a high order beta singularity:

$$B\left(a+i\gamma,\,-a-i\gamma\right)=\int\limits_0^1 t^{a+i\gamma-1}(1-t)^{-a-i\gamma-1}dt. \tag{2}$$

Using the following substitutions

$$\left(\frac{t}{1-t}\right)^{a+i\gamma}=\exp\left(a+i\gamma\right)\xi,\quad \xi=\ln\frac{t}{1-t},\quad d\xi=\frac{dt}{t(1-t)},$$

from the formula (2) one obtains:

$$B\left(a+i\gamma,\,-a-i\gamma\right)=F\left[\exp\left(a\xi\right)\right]\left(\gamma\right), \tag{3}$$

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where $F\left[\exp\left(a\xi\right)\right]\left(\gamma\right)$ denotes the Fourier transform of the function $\exp\left(a\xi\right)$, and

$$F\left[\exp\left(a\xi\right)\right]\left(\gamma\right)=\int\limits_{-\infty}^{+\infty}d\xi\exp\left(i\gamma\xi\right)\exp\left(a\xi\right). \tag{4}$$

Let's decompose the right-hand-side of formula (4):

$$F\left[\exp\left(a\xi\right)\right]\left(\gamma\right)=\lim_{\nu\to\infty}\int\limits_{-\nu}^{+\nu}\exp\left(i\gamma\xi\right)\sum_{0}^{\infty}\frac{1}{n!}(a\xi)^{n}d\xi=\sum_{0}^{\infty}\frac{(a)^{n}}{n!}\lim_{\nu\to\infty}\int\limits_{-\nu}^{+\nu}(\xi)^{n}\exp\left(i\gamma\xi\right)d\xi. \tag{5}$$

Substituting the relation (see, e.g., [2], p. 57)

$$\lim_{\nu\to\infty}\int\limits_{-\nu}^{+\nu}d\xi(i\xi)^{n}\exp\left(i\gamma\xi\right)=2\pi\delta^{(n)}\left(\gamma\right)$$

into (5), after simple transformations, one obtains the following functional:

$$F\left[\exp\left(a\xi\right)\right]\left(\gamma\right)=2\pi\exp\left(-ia\frac{d}{d\gamma}\right)\delta\left(\gamma\right). \tag{6}$$

By the formulas (3) and (6), one gets (for convenience, we introduce the notation):

$$f\left(\gamma-ia\right)\equiv B\left(a+i\gamma,\,-a-i\gamma\right)=2\pi\exp\left(-ia\frac{d}{d\gamma}\right)\delta\left(\gamma\right). \tag{7}$$

Using the expressions (7) and taking into account the next formula (see, e.g., [2], p. 44):

$$\int\limits_{-\infty}^{+\infty}\varphi\left(\gamma\right)\delta^{(n)}\left(\gamma\right)d\gamma=(-1)^{n}\varphi^{(n)}\left(0\right),$$

it is easy to verify, that:

$$\int\limits_{-\infty}^{+\infty}\varphi\left(\gamma\right)f\left(\gamma-ia\right)d\gamma=2\pi\varphi\left(ia\right) \tag{8}$$

for any function $\varphi\left(\gamma\right)$ of the basic space (see, e.g., [3], pp. 15 and 27).

Moreover, since

$$f\left(\gamma-ia\right)=2\pi u\delta\left(\gamma\right),\quad u=\exp\left(-ia\frac{d}{d\gamma}\right),$$

by virtue of the following relations (see, e.g., [2], p. 22)

$$(f(\gamma - ia), \varphi) = 2\pi (u\delta, \varphi) = 2\pi (\delta, \varphi(\gamma + ia)),$$

and by formula (7), one can write:

$$\int_{-\infty}^{+\infty} \varphi(\gamma) B(a + i\gamma, -a - i\gamma) d\gamma = 2\pi \int_{ia-\infty}^{ia+\infty} \varphi(\eta) \delta(\eta - ia) d\eta = 2\pi\varphi(ia). \tag{9}$$

From expression (9) one can conclude that (see, e.g., [2], p. 200):

$$g(\eta - ia) \equiv B(a + i\eta, -a - i\eta) = 2\pi\delta(\eta - ia). \tag{10}$$

2 The analytical representation. Since the functions (10) are finite functions, by the Cauchy formula of analytic continuation, we obtain (see, e.g., [4], p. 468):

$$B(a + iz, -a - iz) = \frac{1}{i} \int_{-\infty+ia}^{+\infty+ia} \delta(\eta - ia) K(\eta - z) d\eta = -\frac{1}{i} \frac{1}{z - ia}, \tag{11}$$
$$z = x + iy, \; y \neq a,$$

where $K(\eta - z)$ is the Cauchy kernel.

Since (11) represents a holomorphic function in the whole complex plane, except the point $z = ia$, it can be continued analytically everywhere except this point.

Thus, by formulas (10) and (11), one can write:

$$g(z - ia) = B(a + iz, -a - iz) = 2\pi\delta(z - ia) = -\frac{1}{i} \frac{1}{z - ia}, \tag{12}$$
$$z = x + iy, \; z \neq ia.$$

Since (12) is holomorphic on the upper and lower open half-planes that are divided by the line $\eta = x + ia$, for any function $\varphi(\eta)$ from the area of basic functions the following limit exists (see, e.g., [2], p. 28, [4], p. 469):

$$\lim_{y\to a} \int_{-\infty+ia}^{+\infty+ia} \varphi(\eta) g(\eta + iy - 2ia) d\eta = -\frac{1}{i} \lim_{y\to a} \int_{-\infty+ia}^{+\infty+ia} \frac{\varphi(\eta)}{\eta + iy - 2ia} d\eta$$
$$= -\frac{1}{i} \lim_{\varepsilon\to 0} \int_{-\infty+ia}^{+\infty+ia} \frac{\varphi(\eta)}{\eta - ia + i\varepsilon} d\eta. \tag{13}$$

From relations (12) and (13), one obtains the limiting behavior of the beta functional:

$$\lim_{y\to a} \int_{-\infty+ia}^{+\infty+ia} \varphi(\eta) B(a + iz, -a - iz) d\eta = -\frac{1}{i} \lim_{\varepsilon\to 0} \int_{-\infty+ia}^{+\infty+ia} \frac{\varphi(\eta)}{\eta - ia + i\varepsilon} d\eta, \tag{14}$$
$$z = \eta + iy - ia,$$

where the generalized function, entering in (14) is defined as follows:

$$\lim_{\varepsilon\to 0}\int\limits_{-\infty+ia}^{+\infty+ia}\frac{\varphi(\eta)}{\eta-ia+i\varepsilon}d\eta=-i\pi\varphi(ia)+P\int\limits_{-\infty+ia}^{+\infty+ia}\frac{\varphi(\eta)}{\eta-ia}d\eta.$$

3 Conclusions. The Euler Integral of the first kind is studied and it is shown that in the region of its divergence for some pairs of parameters it has a functional character. The functional form (7) of the beta-integral is found, its analytical representation and asymptotic expression are derived by (12), (14).

The example of the beta integral shows, that some analytical function and generalized function have the common origin.

R E F E R E N C E S

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