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GENERATING FUNCTIONS AND SPECTRAL ASYMPTOTICS OF SELF-SIMILAR FRACTAL STRINGS

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Abstract. The geometric zeta function associated with a fractal string captures some of its essential features and in particular the intrinsic oscillations in its geometry and spectrum. Using a generating functions approach, we show how to obtain the asymptotic behaviour of the spectral counting function of self-similar fractal strings by elementary methods.

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1 Introduction and definitions. In a previous paper [1], we investigated the modified one-dimensional Weyl-Berry conjecture using elementary methods and discussed the remarkable dichotomy between Minkowski measurable and non Minkowski measurable fractal strings. In this work, we propose a more detailed analysis of the important subclass of self-similar strings.

Let Ω be a bounded domain in $\mathbb{R}$ with boundary $\delta\Omega$. We consider the eigenvalue problem:

$$-\Delta u = \lambda u,$$

with Dirichlet boundary conditions, i.e. $u\,|_{\delta\Omega} = 0$. Its set of eigenvalues, $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ - each eigenvalue being repeated according to (algebraic) multiplicity - is countable and the eigenvalue counting function may be defined as:

Definition 1. Eigenvalue counting function.
For a given positive λ, the eigenvalue counting function $N(\lambda)$ is defined as the number of eigenvalues less than λ:

$$N(\lambda) := \#\{(0 <)\lambda_j < \lambda\}.$$

Remark 1. Modified Weyl-Berry conjecture.
The modified Weyl-Berry conjecture for the asymptotics of the eigenvalues of the Laplacian on bounded open subsets of the line (fractal strings) states that:

$$N(\lambda) = \pi^{-1}\,|\Omega|_1\,\lambda^{\frac{1}{2}} + \mathcal{O}(\lambda^{\frac{d}{2}}),$$

with $|\Omega|_1$ being the one-dimensional Lebesgue measure of Ω and $d_M \in [0,1]$ the Minkowski dimension of the boundary.

Definition 2. Ordinary fractal strings.
An ordinary fractal string $\mathcal{L}$ is a one-dimensional drum with fractal boundary $\delta\mathcal{L}$. In other words, it is a nonempty bounded open subset of $\mathbb{R}$, consisting of countably many pairwise disjoint open intervals, called lengths of the string. Here, the listing order of the lengths is irrelevant such that we may write $\mathcal{L} = \{\ell_k\}_{k=1}^{\infty} = \{l_n : l_n \text{ has multiplicity } \eta_n\}_{n=1}^{\infty}$.

Definition 3. Self-similar sets.
Given $N \geq 2$ contraction similitudes $\Phi_j : [0,1] \to [0,1]$, with scaling ratios $0 < r_j < 1$: $|\Phi_j(x) - \Phi_j(y)| = r_j|x-y|, \forall x, y \in [0,1]$. Then, the unique nonempty compact subset F of $[0,1]$ satisfying the fixed point equation: $F = \bigcup_{j=1}^{N} \Phi_j(F)$ is called a self-similar set.

Definition 4. Open the set condition.
The system of maps $\Phi := \{\Phi_j : j = 1, ..., N\}$ satisfies the open set condition if there exists a nonempty open subset U of $[0,1]$, such that $\Phi_j(U) \cap \Phi_{j'}(U) = \emptyset$, for all $j \neq j'$, $j, j' \in \{1, ..., N\}$, and $\Phi_j(U) \subset U, \forall j \in \{1, ..., N\}$.

Definition 5. Self-similar fractal string.
A self-similar set F satisfying the open set condition determines a self-similar string $\mathcal{L}$ as: $\mathcal{L} := [0,1] \setminus F,$ with boundary $\delta\mathcal{L} = F$.

Definition 6. Lattice vs. Non-lattice.
Let Φ be a self-similar system with scaling ratios $(r_j)_{j=1}^{N}$ and attractor F. The associated string $\mathcal{L}$ is said to be lattice if there exist $0 < r < 1$ and N positive integers k_j, such that $\gcd(k_1, \ldots, k_N) = 1$ and $r_j = r^{k_j}$ for $j = 1, \ldots, N$. Otherwise, the associated string $\mathcal{L}$ is called non-lattice.

Definition 7. The geometric zeta-function.
The geometric zeta function of a fractal string $\mathcal{L}$ is defined as:

$$\zeta_{\mathcal{L}}(s) := \sum_{n=1}^{\infty} \eta_n l_n^s,$$

for $Re(s) > d_M$, where d_M is the Minkowski dimension of the boundary $\delta\mathcal{L}$ of the string.

Remark 2. The geometric zeta-function of a self-similar string.
For a self-similar string $\mathcal{L}$, the geometric zeta function is given by:

$$\zeta_{\mathcal{L}}(s) = \frac{1}{1 - \sum_{j=1}^{N} r_j^s},$$

where the r_j are the scaling ratios of the contraction similitudes.

Remark 3. The geometric zeta-function of a self-similar string as a generating function.
Some values of the geometric zeta function of a string $\mathcal{L}$ have a special interpretation. In particular, for $s = 1$, we have

$$\zeta_{\mathcal{L}}(1) = \sum_{n=1}^{\infty} \eta_n l_n = \frac{1}{1 - \sum_{j=1}^{N} r_j}$$

for a self-similar string, such that the right-hand side may be viewed as a generating function for the multiplicities of lengths.

Remark 4. The Minkowski dimension of a self-similar string.
The Minkowski dimension d_M of a self-similar string $\mathcal{L}$ may be obtained by solving the Moran equation:

$$1 - \sum_{j=1}^{N} r_j^{d_M} = 0, \text{ for } d_M.$$

2 Results. The following propositions clearly demonstrate the difference in the spectral behaviour of lattice and non-lattice fractal strings. Although these results have already been obtained previously (see for example [2]) using the theory of complex dimensions, the approach proposed here appears more amenable and tractable.

Proposition 1. *The eigenvalue counting function $N(\lambda)$ of a lattice self-similar string with $0 < d_M < 1$ never admits a monotonic asymptotic second term.*

Proof. In the lattice case, the geometric zeta function at 1 is of the form:

$$\zeta_{\mathcal{L}}(1) = \frac{1}{1 - \sum_{j=1}^{N} a_j r^j}$$

with its Taylor-expansion given by:

$$\zeta_{\mathcal{L}}(1) = \sum_{k=1}^{\infty} \eta_k r^k = \sum_{k=1}^{\infty} \eta_k l_k,$$

where $l_k = r^k$. In general, the asymptotic growth of η_k is given by $\eta_k \approx \rho^{-k}$, where ρ is the smallest root of $1 - \sum_{j=1}^{N} a_j r^j = 0$ (Cauchy-Hadamard theorem). Thus,

$$\lim_{n\to\infty} \frac{\sum\limits_{k=n}^{\infty} \eta_k l_k}{\sum\limits_{k=n+1}^{\infty} \eta_k l_k} = \lim_{n\to\infty} \frac{\frac{r^n}{\rho^{n-1}(\rho - r)}}{\frac{r^{n+1}}{\rho^n(\rho-r)}} = \frac{\rho}{r}.$$

By Moran's equation, we have $\rho = r^{d_M}$, and therefore: $\frac{\rho}{r} = r^{d_M - 1}$, as $r < \rho = r^{d_M}$, for $0 < r < 1$ and $0 < d_m < 1$. Using now the criterion from [1], the proof is completed, as we always have $r^{d_M - 1} \neq 1$ under these conditions. □

Proposition 2. *The eigenvalue counting function $N(\lambda)$ of a non-lattice self-similar string with $0 < d_M < 1$ always admits a monotonic asymptotic second term.*

Proof. In [2], theorem 3.18, it has been established that every nonlattice string can be approximated by a sequence of lattice strings. For the sake of simplicity, but without loss of generality, we will here consider a non-lattice string with only two different scaling ratios r_1, r_2 and weights a_1, a_2, the results obtained will remain true in the general case. The geometric zeta function at 1 is then of the form:

$$\zeta_{\mathcal{L}}(1) = \frac{1}{1 - a_1 r_1 - a_2 r_2} = \frac{1}{1 - a_1 r_1 - a_2 r_1^{\frac{\ln(r_2)}{\ln(r_1)}}}$$
$$:= \frac{1}{1 - a_1 r_1 - a_2 r_1^{\gamma}}, \text{ with } \gamma := \frac{\ln(r_2)}{\ln(r_1)}.$$

Using the Diophantine approximation for the exponent γ, i.e.: $\gamma = lim_{n\to\infty} \frac{p_n}{q_n}$, where $(p_n, q_n) \in \mathbb{N}^2$ and the sequences $(p_n)_{n\in\mathbb{N}}$, resp. $(q_n)_{n\in\mathbb{N}}$ are strictly increasing towards infinity, we can write:

$$\zeta_{\mathcal{L}}(1) = \lim_{n\to\infty} \frac{1}{1 - a_1 r^{q_n} - a_2 r^{p_n}}, \text{ with } r := r_1^{\frac{1}{q_n}}.$$

Now, as $lim_{n\to\infty} q_n = \infty$:

$$\lim_{n\to\infty} \frac{\sum\limits_{k=n}^{\infty} \eta_k l_k}{\sum\limits_{k=n+1}^{\infty} \eta_k l_k} = \lim_{n\to\infty} \frac{\frac{r^n}{\rho^{n-1}(\rho-r)}}{\frac{r^{n+1}}{\rho^n(\rho-r)}} = \frac{\rho}{r} = r^{d_M-1} = r_1^{\frac{d_M-1}{q_n}} = 1,$$

which completes the proof. $\square$

3 Conclusions. The asymptotic behaviour of the eigenvalue counting functions of self-similar strings was obtained by elementary methods. However, as the presence of oscillations is closely related to Riemann's conjecture ([3],[4]), it would be highly desirable to apply the approach proposed in [1] to fractal strings that are not self-similar. We expect to present several more extensive results on the subject in a subsequent paper.

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