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ON THE EXACT SOLUTION OF THE ROTATING THREE-AXIS GAS ELLIPSOID OF JACOBI WHICH IS IN ITS OWN GRAVITATIONAL FIELD

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Abstract. In this work the exact solution of a problem on stationary solid-state rotation with a constant angular speed of a homogeneous three-axis gas ellipsoid (the law and speed of the movement of the medium and also thermodynamic characteristics of the medium) which is in its own gravitational field and adjoining on emptiness is found (zero pressure on boundary). The exact distribution of gravitational potential in a three-axis homogeneous ellipsoid satisfying Poisson's equation is also found.

Keywords and phrases: Rotating three-axis gas ellipsoid of Jacobi, gravitational field, exact solution.

AMS subject classification (2010): 97M50, 83F05, 76U05.

1 Introduction. The mathematical modeling of astrophysics processes are one of the most actual problems of modern applied mathematics [1-6]. To resolve a number of astrophysical problems one has to investigate the dynamics of the gas bodies that interact with a gravitating field. It is clear that the conceptions of astrophysical problems investigation can be based on the statement and solution of a number of gas motion dynamic problems. These problems are regarded as theoretic models that include important peculiarities of the motion and evolution of stars.

2 Statement of the mixed problem for the system of equations in partial derivatives. At the solution of various problems connected with the movement of gravitating gas in gas bodies of various configuration (a sphere, Makloren's spheroid, a three-axis ellipsoid of Jacobi, etc.) it is convenient to use spherical coordinates:

$$\frac{du_r}{dt} - \frac{u_\theta^2 + u_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial \Phi}{\partial r}, \tag{1}$$

$$\frac{du_\theta}{dt} + \frac{u_r u_\theta}{r} - ctg\theta\frac{u_\varphi^2}{r} = -\frac{1}{r\rho}\frac{\partial p}{\partial \theta} + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}, \tag{2}$$

$$\frac{du_\varphi}{dt} + \frac{u_r u_\varphi}{r} + ctg\theta\frac{u_\theta u_\varphi}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \frac{1}{r}\frac{\partial \Phi}{\partial \varphi}, \tag{3}$$

$$r^2 \sin\theta\frac{\partial \rho}{\partial t} + \sin\theta\frac{\partial}{\partial r}(\rho u_r r^2) + r\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) + r\frac{\partial \rho u_\varphi}{\partial \varphi} = 0, \tag{4}$$

$$\frac{d}{dt}\Big(\frac{p}{\rho^\gamma}\Big) = 0, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}, \tag{5}$$

where $p(r,\theta,\varphi,t)$ is pressure, $\rho(r,\theta,\varphi,t)$ is density, γ is the specific heat ratio, $\vec{u}(u_r,u_\theta,u_\varphi)$ are physical components of a vector of speed of the movement of particles of the medium.

$$u_r = \frac{dr}{dt}, \quad u_\theta = r\frac{d\theta}{dt}, \quad u_\varphi = r\sin\theta\frac{d\varphi}{dt}. \tag{6}$$

The potential of the gravitational field in a body satisfies Poisson's equation

$$\Delta\Phi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\Phi}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = -4\pi k\rho_{int}, \tag{7}$$

where k is the gravitational constant.

3 Exact solution of a problem on rotation of a gravitating gas ellipsoid of Jacobi. Let's assume that the homogeneous ($\rho_{int} = const$) three-axis gas ellipsoid which is in its own gravitational field and adjoining on emptiness with a constant angular ω speed solid-state rotates around an z axis.

It is necessary to find the speed and the law of the movement and also thermodynamic characteristics of the medium i.e. to solve the system of the equations in partial derivatives (1) - (7) with a zero boundary condition for pressure upon ellipsoid surfaces.

The solution of Poisson's equation (7) gives distribution of gravitational potential in a uniform three-axis ellipsoid [7].

$$\Phi(x,y,z) = \pi k\rho abc\int_0^\infty (1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{c^2+s})\frac{ds}{\Gamma}, \tag{8}$$

$$\Phi(r,\theta,\varphi) = \Phi_0 - Pr^2\sin^2\theta\cos^2\varphi - Qr^2\sin^2\theta\sin^2\varphi - Rr^2\cos^2\theta, \tag{9}$$

$$\Phi_0 = \pi k\rho abc\int_0^\infty \frac{ds}{\Gamma}, \quad P = \pi k\rho abc\int_0^\infty \frac{1}{a^2+s}\frac{ds}{\Gamma}, \tag{10}$$

$$Q = \pi k\rho abc\int_0^\infty \frac{1}{b^2+s}\frac{ds}{\Gamma}, \quad R = \pi k\rho abc\int_0^\infty \frac{1}{c^2+s}\frac{ds}{\Gamma}, \quad \Gamma(s) = \sqrt{(a^2+s)(b^2+s)(c^2+s)}$$

where a,b,c are half shafts of a three-axis ellipsoid of rotation, x,y,z are cartesian coordinates, ρ is constant density of a gas ellipsoid.

In spherical r,θ,φ coordinates distribution of speed of the movement of gravitating gas in the homogeneous three-axis ellipsoid which is in its own gravitational field and solid-state rotating with a ω constant angular speed around an z axis has the form:

$$u_r = u_\theta = 0, \ u_\varphi = r\sin\theta\frac{d\varphi}{dt}, \ \frac{d\varphi}{dt} = \omega = const. \tag{11}$$

Substituting (9) - (11), in (1) - (3), we will get

$$\frac{\partial p}{\partial r} = \rho r\left[\omega^2\sin^2\theta - 2\left(P\sin^2\theta\cos^2\varphi + Q\sin^2\theta\sin^2\varphi + R\cos^2\theta\right)\right], \tag{12}$$

$$\frac{\partial p}{\partial \theta} = \rho r^2 \left[\frac{\omega^2}{2} - P \cos^2 \varphi - Q \sin^2 \varphi + R \right] \sin 2\theta, \tag{13}$$

$$\frac{\partial p}{\partial \varphi} = \rho r^2 (P - Q) \sin^2 \theta \sin 2\varphi. \tag{14}$$

Let's note that the equations of continuity (4) and adiabaticity (5) owing to constancy of density and (11) are carried out automatically.

The three-axis ellipsoid (Jacobi's ellipsoid) in the spherical coordinates has the form:

$$\left[\frac{r}{a}\right]^2 = \frac{1}{\sin^2 \theta \left[1 + I_1^2 \sin^2 \varphi\right] + (I_2^2 + 1) \cos^2 \theta}, \tag{15}$$

$$a > b, \; a > c, \quad I_1^2 = \frac{a^2 - b^2}{b^2}, \quad I_2^2 = \frac{a^2 - c^2}{c^2}.$$

As on boundary (15) with emptiness (vacuum) pressure of gravitating gas is equal to zero, the exact solution of a boundary problem (12) - (14) i.e. distribution of pressure gravitating gas in Jacobi's ellipsoid has the following form

$$p(r, \theta, \varphi) = \frac{\rho}{2} \left[\omega^2 \sin^2 \theta - 2(P \sin^2 \theta \cos^2 \varphi + Q \sin^2 \theta \sin^2 \varphi + R \cos^2 \theta)\right] A, \tag{16}$$

$$A \equiv r^2 - \frac{a^2}{\sin^2 \theta (1 + I_1^2 \sin^2 \varphi) + (I_2^2 + 1) \cos^2 \theta},$$

at the same time the ω angular speed of rotation has to satisfy a ratio

$$\omega^2 = \frac{2}{I_1^2} \left[P(I_1^2 + 1) - Q\right] \tag{17}$$

and half shafts of a steady three-axis ellipsoid of rotation of Jacobi, have to satisfy the following ratios:

$$(1 + I_2^2)(Q - P) = I_1^2 R, \tag{18}$$

$$a > b > c, \quad \frac{1}{c^2} > \frac{1}{a^2} + \frac{1}{b^2} \left(I_2^2 > I_1^2 + 1\right), \quad \frac{Q}{I_1^2 + 1} < P < Q < R < \frac{I_2^2 + 1}{I_1^2 + 1} Q.$$

The ratio (18) in the theory of ellipsoidal figures of rotation is called Jacobi's formula.

4 Conclusions. The exact solution of a problem on rotation of the gravitating gas ellipsoid of Jacobi adjoining on a vacuum is found. Further, by solving the mixed problem on the movement of a blast shock wave in a gravitating gas ellipsoid of Jacobi, the boundary condition on a surface of a rupture of the first kind of solutions will be known.

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