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PROBLEM OF STATICS OF THE LINEAR THERMOELASTICITY OF THE MICROSTRETCH MATERIALS WITH MICROSTRUCTURE AND MICROTEMPERATURES

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Abstract. The representation formula of a general solution of the homogeneous system of differential equations obtained in the paper is expressed by means of three harmonic and four metaharmonic functions. These formulas are very convenient and useful in many particular problems for domains with concrete geometry.

Keywords and phrases: Microstretch materials, metaharmonic, microtemperatures.

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1 Introduction. One of the basic methods of solving two- and three-dimensional problems of rigid deformable bodies is the Fourier method which is based on the solution of differential equations of a given model by the method of separation of variables in a certain system of curvilinear coordinates. In case the construction of a system of differential equations turns out complicated, its solution can be represented by a simple solution of Laplace and Helmholtz equations. Representations proposed by W. Kelvin, J. Hadamard, J. Boussinesq, M. Papkovich, G. Neuber, E. Trefftz, G. Kolosov, N. Muskhelishvili and other authors are well known in the literature.

2 Basic equations and fundamental theorem. The homogeneous system of static equation of the thermoelasticity theory of microstretch materials with microtemperatures and microdilatation in the case of isotropic bodies read as [1]

$$(\mu+\varkappa)\Delta u+(\lambda+\mu)\operatorname{grad}\operatorname{div}u-\varkappa\operatorname{rot}\omega+\mu_0\operatorname{grad}v-\beta_0\operatorname{grad}\theta=0, \tag{1}$$

$$\varkappa_6\Delta w-\varkappa_2 w+(\varkappa_4+\varkappa_5)\operatorname{grad}\operatorname{div}w-\varkappa_3\operatorname{grad}\theta=0, \tag{2}$$

$$\gamma\Delta\omega-2\varkappa\omega+\varkappa\operatorname{rot}u-\mu_1\operatorname{rot}w=0, \tag{3}$$

$$a_0\Delta v-\eta v-\mu_0\operatorname{div}u-\mu_2\operatorname{div}w+\beta_1\theta=0, \tag{4}$$

$$\varkappa_7\Delta\theta+\varkappa_1\operatorname{div}w=0, \tag{5}$$

where γ, λ, μ, $\varkappa$, η, β_0, β_1, μ_0, μ_1, μ_2, a_0, $\varkappa_j$, $j=1,2,\dots,7$ are the real constants characterizing the mechanical and thermal properties of the body, Δ is the Laplace operator, $u=(u_1,u_2)^\top$ is the displacement vector, $w=(w_1,w_2)^\top$ is the microtemperature vector, ω is the microrotation function, v is the microdilatation function, θ is the temperature, measured from a fixed absolute temperature T_0 ($T_0>0$), the symbol $(\cdot)^\top$ denotes transposition operation, here

$$\operatorname{rot} := \Big(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\Big)^\top, \qquad \operatorname{rot}\omega := \Big(-\frac{\partial \omega}{\partial x_2}, \frac{\partial \omega}{\partial x_1}\Big)^\top,$$

$$\operatorname{rot} u := \operatorname{rot}\cdot u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \qquad \operatorname{rot} w := \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}.$$

The following theorem is valid.

Theorem. *A vector $U = (u, w, \omega, v, \theta)^\top \in C^2(\Omega)$ is a general solution of the homogeneous system (1)–(5) in a domain $\Omega \subset R^2$ if and only if is representable in the form*

$$\begin{aligned} u(x) &= a_4 \operatorname{grad}(r^2\phi_0(x)) + \operatorname{grad}\phi_1(x) + a_6 \operatorname{grad} r^3 \frac{\partial}{\partial r}\phi_2(x) \\ &\quad -2\operatorname{grad}(r^2\phi_2(x)) + 4x\Big(r\frac{\partial}{\partial r}+1\Big)\phi_2(x) + \frac{a_3}{\lambda_1^2}\operatorname{grad}\phi_3(x) \\ &\quad -\frac{\mu_0}{\lambda_0\lambda_4^2}\operatorname{grad}\phi_4(x) + a_7 \operatorname{rot}\phi_5(x) + \frac{\varkappa}{(\mu+\varkappa)\lambda_2^2}\operatorname{rot}\phi_6(x) + \chi^\top(x), \end{aligned} \tag{6}$$

$$w(x) = -\frac{4\varkappa_3\lambda_0}{\varkappa_2}\operatorname{grad}\Big(r\frac{\partial}{\partial r}+1\Big)\phi_0(x) - \lambda_0\varkappa_7\operatorname{grad}\phi_3(x) + \frac{1}{\lambda_3^2}\operatorname{rot}\phi_5(x), \tag{7}$$

$$\omega(x) = 2\operatorname{rot}\Big[x\Big(r\frac{\partial}{\partial r}+1\Big)\phi_2(x)\Big] + \frac{\mu_1}{\gamma(\lambda_3^2-\lambda_2^2)}\phi_5(x) + \phi_6(x) + \chi'(x), \tag{8}$$

$$\begin{aligned} v(x) &= -\frac{4a_1}{\lambda_4^2}\Big(r\frac{\partial}{\partial r}+1\Big)\phi_0(x) - \frac{4\mu_0(a_6+1)}{\eta}r\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}+1\Big)\phi_2(x) \\ &\quad + \frac{a_2}{\lambda_1^2-\lambda_4^2}\phi_3(x) + \phi_4(x) + \chi''(x), \end{aligned} \tag{9}$$

$$\theta(x) = 4\lambda_0\Big(r\frac{\partial}{\partial r}+1\Big)\phi_0(x) + \varkappa_1\lambda_0\phi_3(x), \tag{10}$$

where $\Delta\phi_j(x) = 0$, $j = 0, 1, 2$, $(\Delta-\lambda_1^2)\phi_3(x) = 0$, $(\Delta-\lambda_2^2)\phi_6(x) = 0$, $(\Delta-\lambda_3^2)\phi_5(x) = 0$, $(\Delta-\lambda_4^2)\phi_4(x) = 0$,

$$\lambda_1^2 = \frac{\varkappa_2\varkappa_7 - \varkappa_1\varkappa_3}{l_0\varkappa_7}, \quad \lambda_2^2 = \frac{\varkappa(2\mu+\varkappa)}{\gamma(\mu+\varkappa)}, \quad \lambda_3^2 = \frac{\varkappa_2}{\varkappa_6}, \quad \lambda_4^2 = \frac{\lambda_0\eta - \mu_0^2}{\lambda_0 a_0},$$

$$l_0 = \varkappa_4 + \varkappa_5 + \varkappa_6, \quad \lambda_0 = \lambda + 2\mu + \varkappa, \quad a_1 = \frac{\beta_0\mu_0 - \lambda_0\beta_1}{a_0},$$

$$a_2 = \varkappa_1 a_1 - \frac{\mu_2\varkappa_7\lambda_0\lambda_1^2}{a_0}, \quad a_3 = \varkappa_1\beta_0 - \frac{\mu_0 a_2}{\lambda_0(\lambda_1^2-\lambda_4^2)}, \quad a_4 = \frac{\eta\beta_0 - \mu_0\beta_1}{a_0\lambda_4^2},$$

$$a_5 = \lambda_0\frac{2\mu_0^2(\mu+\varkappa) - \varkappa\lambda_0\eta}{(2\mu+\varkappa)(\lambda_0\eta-\mu_0^2)}, \quad a_6 = \frac{\eta(2\mu+\varkappa)}{2(\lambda_0\eta-\mu_0^2)} - 1, \quad a_7 = \frac{\mu_1\varkappa}{\gamma(\mu+\varkappa)(\lambda_3^2-\lambda_2^2)\lambda_3^2},$$

$$r\frac{\partial}{\partial r} = x\cdot\operatorname{grad}, \quad r = |x|, \quad x = (x_1, x_2), \quad \widetilde{x} = (-x_2, x_1).$$

$$\chi(x) = \begin{cases} A_0x + B_0\widetilde{x}, & x \in \Omega' \\ \frac{1}{r^2}(A_0x + B_0\widetilde{x}), & x \in \Omega'' \end{cases} \quad \chi'(x) = \begin{cases} B_0, & x \in \Omega' \\ 0, & x \in \Omega'', \end{cases} \tag{11}$$

$$\chi''(x) = \begin{cases} -\frac{2\mu_0}{\eta} A_0, & x \in \Omega' \\ 0 & x \in \Omega'', \end{cases}$$

Ω' is a finite two-dimensional region, $\Omega'' = R^2 \backslash \overline{\Omega}'$.

3 Solution of the boudery value problems. Let $\Omega^+ \subset R^2$ be a circle bounded by the circvmference $\partial\Omega$, centered at the origin and having radius R. We denote $\Omega^- := R^2 \backslash \overline{\Omega}^+$.

Problem. Find, in the domain Ω^+ (Ω^-), such a regular vector $U = (u, w, \omega, v, \theta)^\top \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega}^\pm)$ that satisfies in this domain the system of differential equations (1)–(5) and, on the boundary $\partial\Omega$, satisfies one of the following boundary conditions:

(I)$^\pm$ **(Dirichlet problem)**

$$\{U(z)\}^\pm = f(z), \qquad z \in \partial\Omega, \tag{12}$$

(II)$^\pm$ **(Neumann problem)**

$$\{P(\partial, n)U(z)\}^\pm = f(z), \qquad z \in \partial\Omega, \tag{13}$$

where $f = (f^{(1)}, f^{(2)}, f_3, f_4, f_5)^\top$, $f^{(j)} = (f_1^{(j)}, f_2^{j})^\top$, $j = 1, 2$, $f_k^{(j)}$, $k, j = 1, 2$, f_l, $l = 3, 4, 5$ are the functions given on the boundary $\partial\Omega$, $n(z)$ is the outward normal unit vector passing at a poind $z \in \partial\Omega$ with respect to the domain Ω^+, and $P(\partial, n)U$ is the generalized thermo-stress vector.

In the case of the exterior problems for the domain Ω^- the vector $U = (u, w, \omega, v, \theta)^\top$ should satisfy the following decay conditions at infinity

$$U(x) = O(|x|^{-1}), \quad \partial_k U(x) = o(|x|^{-1}), \quad k = 1, 2.$$

We seek the solutions of Dirichlet and Neumann problems by formulas (6)–(10), where

1. For $x \in \Omega^+$

$$\phi_j(x) = a_0^{(j)} + \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k \big(a_k^{(j)} \cos k\varphi + b_k^{(j)} \sin k\varphi\big), \quad j = 0, 1, 2,$$

$$\phi_l(x) = \sum_{k=0}^{\infty} g_k(\lambda_j r) \big(a_k^{(l)} \cos k\varphi + b_k^{(l)} \sin k\varphi\big),$$

$l = 3$, $j = 1$; $l = 4$, $j = 4$; $l = 5$, $j = 3$; $l = 6$, $j = 2$.

2. For $x \in \Omega^-$

$$\phi_j(x) = a_0^{(j)} + \sum_{k=1}^{\infty} \left(\frac{R}{r}\right)^k \big(a_k^{(j)} \cos k\varphi + b_k^{(j)} \sin k\varphi\big), \quad j = 0, 1, 2,$$

$$\phi_l(x) = \sum_{k=0}^{\infty} h_k(\lambda_j r) \big(a_k^{(l)} \cos k\varphi + b_k^{(l)} \sin k\varphi\big),$$

$l=3$, $j=1$; $l=4$, $j=4$; $l=5$, $j=3$; $l=6$, $j=2$. Here

$$g_k(\lambda_j r)=\frac{I_k(\lambda_j r)}{I_k(\lambda_j R)},\quad h_k(\lambda_j r)=\frac{K_k(\lambda_j r)}{K_k(\lambda_j R)},\quad j=3,4,5,6,$$

$a_0^{(j)}$, $a_k^{(j)}$, $b_k^{(j)}$, $j=0,1,\dots,6$ are sought constants, $I_k(x)$ is the Bessel function, $K_k(x)$ is the MacDonald function.

Solutions of Dirichlet and Neumann problems are obtained in the form of absolutely and uniformly convergent series.

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