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ON THE OF NILPOTENT AND SOLVABLE MR-GROUPS

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Abstract. In the present paper, the central series and series of commutants in MR-groups are introduced. Moreover, various definitions of nilpotency in this category are compared.

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The notion of exponential R-group (R is an arbitarry associative ring with identity 1) was introduced by Lyndon in [1]. Myasnikov and Remeslennikov introduced in [2] a new category of exponential R-groups (MR-groups) as a natural generalization of the notion of R-module to a noncommutative case. Recall the basic definitions (see [1, 2]).

Let $L = \langle \cdot\,, ^{-1}, e\rangle$ be the group language (signature); here, $\cdot$ denotes the binary operation of multiplication, $^{-1}$ denotes the unary operation of inversion, and e is a constant symbol for the identity element of the group.

We enrich the group language to the language $\mathfrak{L}^*_{gr} = \mathfrak{L}_{gr} \cup \{f_\alpha(g) \mid \ \alpha \in R\}$, where $f_\alpha(g)$ is a unary algebraic operation.

Definition 1 ([1]). A Lyndon R-group is a set G on which operations, $\cdot$, $^{-1}$, e and $\{f_\alpha(g) \mid \ \alpha \in R\}$ are defined and the following axioms hold:

(i) the group axioms;

(ii) for all $g, h \in G$ and all elements $\alpha, \beta \in R$,

$$g^1 = g, \ \ g^0 = e, \ \ e^\alpha = e; \tag{1}$$

$$g^{\alpha+\beta} = g^\alpha \cdot g^\beta, \ \ g^{\alpha\beta} = (g^\alpha)^\beta; \tag{2}$$

$$(h^{-1}gh)^\alpha = h^{-1}g^\alpha h. \tag{3}$$

For, brevity, in the formulas expressing the axioms, we write $f_\alpha(g)$ instead of g^α for $g \in G$ and $\alpha \in R$.

Let $\mathfrak{L}_R$ denote the category of all Lyndon R-groups. Since the axioms given above are universal axioms of the language $\mathfrak{L}^*_{gr}$, it follows that $\mathfrak{L}_R$ is a variety of algebraic systems in the language $\mathfrak{L}^*_{gr}$; therefore, general theorems of universal algebra allow us to consider the varieties of R-groups, R-homomorphisms, R-isomorphisms, free R-groups, and so on.

MR**-exponential groups.** There exist Abelian Lyndon R-groups which are not R-modules (see [3], where the structure of a free Abelian R-group was studied in detail). The authors of [1] augmented Lyndon's axioms (quasi-identity):

$$(MR) \qquad \forall_{g,h\in G}, \ \ \alpha \in R \ \ [g,h] = e \Longrightarrow (gh)^\alpha = g^\alpha h^\alpha \ \ \big([g,h] = h^{-1}h^{-1}gh\big). \tag{4}$$

Definition 2 ([2]). An MR-group is a group G on which the operations g^α are defined for all $g \in G$ and $\alpha \in R$ so that axioms (1)–(4) hold.

Let $\mathfrak{M}_R$ denote the class of all R-exponential groups with axioms (1)–(4). In honor of Myasnikov, R-groups with an extra axiom were called in [4] MR-groups (R is a ring). Clearly, this class is a quasi-variety in the language $\mathfrak{L}^*_{gr}$, and free MR-groups, MR-homomorphisms, and so on are defined; moreover, each Abelian MR-group is an R-module and vice versa.

Most of natural examples exponential group belongs to the class $\mathfrak{M}_R$. For example, unipotent groups over a field K of zero characteristic are MK-groups, pro-p-groups are exponential groups over the ring of p-adic integers, etc (see [2] for examples).

A systematic study of MR-group was initiated in [4–11]. Results obtained in these papers have turned out to be very useful in solving well-known problems of Tarski.

Below, following [2], we recall some definitions in the category of MR-groups. Let G be an MR-group.

Definition 3 ([2]). A homomorphism of R-groups $\varphi : G_1 \to G_2$ is called an ***R*-homomorphism** if $\varphi(g^\alpha) = \varphi(g)^\alpha$, $g \in G$, $\alpha \in R$.

Definition 4 ([2]). For $g, h \in G$ and $\alpha \in R$, the element $(g,h)_\alpha = h^{-\alpha} g^{-\alpha} (gh)^\alpha$ is called the α-**commutator** of the elements g and h.

It is obvious that for $\alpha = -1$ the α-commutator $(g,h)_\alpha$ coincides with the usual commutator $[h^{-1}, g^{-1}]$.

Clearly, $(gh)^\alpha = g^\alpha h^\alpha (g,h)_\alpha$ and $G \in \mathfrak{M}_R \iff \big([g,h] = e \implies (g,h)_\alpha = e\big)$. This equivalence leads to the definition of an $\mathfrak{M}_R$-ideal.

Definition 5 ([2]). A normal R-subgroup $H \trianglelefteq G$ is called an $\mathfrak{M}_R$-ideal if, $(g,h)_\alpha \in H$ for all $g \in G$, $h \in H$ and $\alpha \in R$.

Proposition. ([2])

(i) *If $\varphi : G_1 \to G_2$ is an R-homomorphism in the category $\mathfrak{M}_R$-groups, then* $\ker\varphi$ *is an $\mathfrak{M}_R$-ideal in G.*

(ii) *If H is an $\mathfrak{M}_R$-ideal in G, then $G/H \in \mathfrak{M}_R$.*

Nilpotent *R*-groups. Let $c > 1$ be a natural number. Denote by $\mathcal{N}_{c,R}$ the category of nilpotent R-groups of nilpotence c from the class $\mathfrak{L}_R$, i.e. of the R-groups where the identity

$$\forall\, x_1, \ldots, x_{c+1} \;\; [x_1, \ldots, x_{c+1}] = e$$

is fulfilled, and by $\mathcal{N}^0_{c,R}$ the category of nilpotent MR-groups of step c. The structure of R-groups without the axiom of choice (MR) is very complicated and that's why only the MR-group is studied in most of the works. In the rest of this paper only the MR-groups will be considered.

Let G be an arbitrary MR-group. Assume

$$(G,G)_R = \langle (g,h)_\alpha \mid \quad g,h \in G, \quad \alpha \in R \rangle_R.$$

We will call a subgroup $(G,G)_R$ a R-commutant of the group G.

Theorem 1. *For any MR-group G the following statements are true:*

(1) *a R-commutant of G is a verbal MR-subgroup defined by the word $[x,y] = x^{-1}y^{-1}xy$;*

(2) *a R-commutant is the smallest $\mathfrak{M}_R$-ideal by which the factor group is abelian.*

For $G \in \mathfrak{M}_R$, we call a R-commutant $(G,G)_R$ ***the first R-commutant*** and denote it by $G^{(1,R)}$. A R-commutant of $G^{(1,R)}$ is called ***the second R-commutant*** and denoted by $G^{(2,R)}$, and so on. There arises a decreasing series of R-commutants

$$G = G^{(0,R)} \geq G^{(1,R)} \geq \cdots \geq G^{(n,R)} \geq \cdots . \tag{5}$$

Definition 6. An exponential MR-group G is called ***solvable*** of there exists a natural number n such that $G^{(n,R)} = e$.

By induction with respect to n it is easy to show that the ordinary n-th commutant $G^{(n)}$ is contained in $G^{(n,R)}$. Hence an n-step solvable group in the category $\mathfrak{M}_R$ is n-step solvable in the category of groups.

Let us proceed to the definition of the lower central series in the category of power MR-groups. The first member of this series is the R-commutant of the group G which we denote by $G_{(1,R)}$. Assume that the n-th member of the lower central series $G_{(R,R)}$ has already been defined. Then $G_{(R+1,R)} = id([G, G_{(R,R)}])$, i.e. $G_{(R+1,R)}$ is the $\mathfrak{M}_R$-ideal generated by the reciprocal commutant of G and $G_{(R,R)}$. There arises the lower central series

$$G = G_{(0,R)} \geq G_{(1,R)} \geq \cdots \geq G_{(R,R)} \geq \cdots . \tag{6}$$

Definition 7. A lower MR-group will be called l***ower R-nilpotent*** if there exists a natural number n such that $G_{(R,R)} = e$. The smallest number n with such a property is called ***the step of R-nilpotence***.

Since the ordinary member of the lower central series $G_{(R)}$ is contained in $G_{(R,R)}$, the n-step lower nilpotent group in the category $\mathfrak{M}_R$ is a nilpotent group of step $\leq n$ in the category of groups. From the definition of series $(5), (6)$ and the definition of a verbal MR-subgroup it directly follows that for any natural number n and ring R the groups $G^{(n,R)}$ and $G_{(n,R)}$ are verbal MR-subgroups. Hence there arise the following questions.

We denote by $\underline{\mathfrak{N}}_{n,R}$ the class of lower R-nilpotent groups of step n. We also introduce other definitions of nilpotence in the category of step MR-groups. For this, by induction with respect to n we define the notion of a simple $\overline{\alpha}$-commutator of weight n, where $\overline{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$. If $n = 2$, then $\overline{\alpha} = (\alpha)$ is the above-defined α-commutator $(g_1, g_2)_\alpha$ of elements g_1, g_2 from G. Assume that for $n \geq 2$ the simple $\overline{\alpha}$-commutators of weight

n have already been defined. Then a simple $(\overline{\alpha}, \alpha_n)$-commutator is an element $(x, g_n)_{\alpha_n}$, where x is a simple $\overline{\alpha}$-commutator. Further, let $X = \{x_1, x_2, \dots\}$ be the set of letters. Denote by W_n the set $W_n = \{(\cdots((x_1, x_2)_{\alpha_1}, x_3)_{\alpha_2}, \dots, x_{n+1})_{\alpha_n} : \ \alpha_1, \dots, \alpha_n \in R\}$ of all simple $\overline{\alpha}$-commutators of weight $n+1$ of the letters $x_1, \dots, x_n$. Denote by $\mathfrak{N}_{n,R}$ the group manifold defined by the set of R-words W_n. The groups of this manifold are called ***R-nilpotent*** MR-groups of nilpotence step n. We denote by $\overline{\mathfrak{N}}_{n,R}$ the manifold of R-groups defined by the word $v_n = [\cdots[[x_1, x_2], x_3], \dots, x_{n+1}]$. The groups of this manifold are called ***upper nilpotent*** groups of step n. The corresponding verbal MR-subgroup is denoted by $\overline{\mathfrak{N}}_{n,R}$. We obviously have the inclusions $\underline{\mathfrak{N}}_{n,R} \subseteq \mathfrak{N}_{n,R} \subseteq \overline{\mathfrak{N}}_{n,R}$. Let us clarify the nature of these inclusions for small values of n.

Theorem 2. *For $n = 1, 2$, all the three definitions of nilpotence coincide.*

Theorem 3. *If $G \in \mathfrak{N}_{2,R}$, then its tensor completion $G^S \in \mathfrak{N}_{2,S}$.*

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