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ON THE CREATION OF INITIAL DATA BY GAUSS-HERMITE APPROXIMATION METHOD FOR THE CAUCHY PROBLEM

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Abstract. This report represents the part of cycle of works dedicated to the solution of Cauchy problem for evolution equation by high order of accuracy. For the permission of this problem the necessary stage is the problem of approximate solution of boundary value problems for the systems of partial differential equations (PDEs) by high order accuracy. In this report we consider the cases when the object of creation represents 2-dim strong elliptic systems of PDEs in the square e with classical boundary conditions when as examples consider well-known typical DEs, refined theories with variable thickness, the stable hierarchical models corresponding to the thin-walled elastic structures. As mathematical apparatus we used the continuous analogy of Douglas-Rachford alternative direction and multipointing difference methods, operator factorization schemes.

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1. Let us cosider strong elliptic PDEs with boundary conditions (BC)

$$Lu=\left(\frac{\partial}{\partial x_i}\left(k^{ij}\frac{\partial}{\partial x_j}\right)-q\right)u(x_1,x_2)=-f(x_1,x_2), \tag{1}$$

$$[u+\sigma(x_1,x_2)N(u)]_{\partial D}=\varphi(x_1,x_2), \tag{2}$$

where $k^{ij}=\left\|k^{ij}_{\alpha\beta}(x_1,x_2)\right\|^n_{\alpha\beta=1}$ is a symmetric matrix $\forall x_1,x_2\in D$, $N(u)=\|N_\alpha(u)\delta_{\alpha\beta}\|^n_1$, $N_\alpha(u)=\sum_{\beta=1}^n k^{ij}_{\alpha\beta}\frac{\partial u_\beta}{\partial x_i}\cos(\nu,x_j)$, ν is an exterior normal, $q=\{q_1(x_1,x_2),...,q_n(x_1,x_n)\}$ is a diagonal matrix, $\sigma(x_1,x_2)>0$ is a scalar function, $u=(u_1,u_2,...,u_n)^T$ is unknown, $f=(f_1,f_2,...,f_n)^T$, $\varphi=(\varphi_1,\varphi_2,...,\varphi_n)^T$ are given vector-functions.

As it's known [1], the process of approximate solution of BVP (1)-(2) is reduced to the solutions of sequences of BVPs with constant coefficients. We named this method of finding $u(x,y)$ as inner process. Here $-L>0$ presents the linear self-jointed operator with constant coefficients, $z^s=u^s-u$, u^s is approximation for function $u(x,y)$, (,) is a scalar product for functions with homogeneous BC corresponding to (2).

Let us define as an exterior iteration process or we named as continuous analogy of the alternating direction method the following schemas:

$$B_1u^{s+\frac12}=B_{12}u^s+f=F_{s+\frac12},\quad \left[u^{s+\frac12}+\sigma_1N\left(u^{s+\frac12}\right)\right]_{\partial_1D}=\varphi_1(x_2), \tag{3}$$

$$B_2u^{s+1} = rIu^{s+\frac{1}{2}} + (B\_rI)u^s = F_{s+1}, \quad [u^{s+1} + \sigma_2 N\left(u^{s+1}\right)]_{\partial_2 D} = \varphi_2(x_1), \tag{4}$$

where r is an iterative parameter, I- is a unit operator, $\sigma=\sigma_1+\sigma_2, B_i = \left\|(r+\hat{q}^i)I + \delta_{\alpha\beta}\hat{A}^i\right\|_{\alpha\beta=1}^n$, $B_{12} = B_1 + L$, $q^1 + q^2 = q$, $\forall x_1, x_2$, $\hat{q} \geq q_\alpha^i(x_1, x_2)$, $\hat{A}_i = -a_i\frac{d^2}{dx_i^2}$, $a_1 > 0$ is a number, $\delta_{\alpha\beta}$ is Kronecker symbol.

The following theorem with respect to the convergence of iteration process of (3-4) is valid.

Theorem. *Let us true the followings one:*

a) *The domain D-is square:* $0 \leq x_1,\ x_2 \leq 1,\ A_i = -\frac{d^2}{dx_i^2}$;

b) $bB \leq -L < \frac{1}{r}B_1B_2$ $(b > 0,\ 0 < r \leq 1)$, $B = (A_1 + A_2)\|\delta_{\alpha\beta}\|$;

c) $u_\alpha^0 \in C^4(0{,}1; 0{,}1)$ $(\alpha = \overline{1,n})$.

Then the sequence of vector-functions u^s convergences to $u(x, y)$.

Proof. From (3)-(4) it follows that:

$$B_1B_2u^{s+1} = (B_1B_2 + rL)u^s + rf,$$

$$\left[u^{s+1} + \sigma_i N\left(u^{s+1}\right)\right]_{\partial_i D} = \varphi_i, \quad \sigma = \sigma_1 + \sigma_2.$$

When the operator $(B_1B_2)^{-1}$ exists for the vector-functions coordinates $\in C^4(0{,}1; 0{,}1)$ it follows that

$$u^{s+1} = (B_1B_2)^{-1}\left[(B_1B_2 + rL)u^s + rf\right], \quad \left[u^{s+1} + \sigma_i N(u)\right]_{\partial_i D} = \varphi_i.$$

Then for the error-vector we have

$$z^{s+1} = (B_1B_2)^{-1}\left[(B_1B_2 + rL)z^s\right], \quad \left[z^{s+1} + \sigma_i N(z^{s+1})\right]_{\partial_i D} = 0,$$

or

$$z^{s+1} = T^s z^0, \quad [z^{s+1} + \sigma_i N\left(z^{s+1}\right)]_{\partial_i D} = 0, \tag{5}$$

where $T^s = [(B_1B_2)^{-1}(B_1B_2 + rL)]^s$.

From condition b) we have

$$B_1B_2 = rbB + C, \quad C > 0, \tag{6}$$

and the inequality:

$$(Tz, z) > 0, \quad \left[z^{s+1} + \sigma_1 N\left(z^{s+1}\right)\right]_{\partial_i D} = 0.$$

On the other hand from condition b), identity (6) and theorem 2 (4.V) of [2] we have

$$(Tz, z) = (Iz, z) - r\left(-L(B_1B_2)^{-\frac{1}{2}}z, (B_1B_2)^{-\frac{1}{2}}\right) \leq (Iz, z)$$

$$-rb\left(B(B_1B_2)^{-\frac{1}{2}}z, (B_1B_2)^{-\frac{1}{2}}z\right) = (Iz, z) - \left((B_1B_2 - C)(B_1B_2)^{-\frac{1}{2}}z, (B_1B_2)^{-\frac{1}{2}}z\right) < \|z\|^2.$$

□

From this inequality it follows that $\|T\| < 1$. Now from (5) and a condition c) $z^{s+1} \to 0$.

2. Let us consider some examples when $N(u) = 0$.

2.1. For the system of the planar theory of elasticity [3] the condition z^{s+1} is satisfied since $b = \mu$, $B = (A_1 + A_2)\|\delta_{jk}\|^2$;

2.2. In case of the thin-walled shell of [4] when $N = 1$ condition b) is satisfied if $b = \min\left\{1 - \varepsilon, \frac{(1-\eta)h^2}{2}\right\}$, $B = (A_1 + A_2)\|\delta_{jk}\|^2$, $B = (A_1 + A_2)\|\delta_{jk}\|_1^s$, $B_i = ((r + 2\mu)I + (2\lambda + 3\mu)A_i)\|\delta_{jk}\|$, $r \in (0,1)$, $0 < \eta < 1$, $(1 + 2h^2\eta)^{-1} < \varepsilon < 1$;

2.3. Let us consider the following PDE:

$$Lu = \frac{\partial}{\partial x_1}\left(k_1(x_1,x_2)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2(x_1,x_2)\frac{\partial u}{\partial x_2}\right) = -(A_1 + A_2)u - qu = f,$$

$$(k_1, k_2 > 0,\ q > 0).$$

In such cases we use both an inner and exterior iterative processes. Instead (3) now we use such schemes:

$$((r + q_1)I + A_1)u^{s+\frac{1}{2}} = (rI - A_2)u^s + f = F_{s+\frac{1}{2}}, \quad \left[u^{s+\frac{1}{2}} + \sigma_1\left(\frac{\partial u^{s+\frac{1}{2}}}{\partial y}\right)\right]_{\partial_1 D} = \varphi_1, \quad (7a)$$

$$((r + q_2)I + A_2)u^{s+1} = (rI - A_1)u^{s+\frac{1}{2}} + f = F_{s+1}, \quad \left[u^{s+1} + \sigma_2\left(\frac{\partial u^{s+1}}{\partial y}\right)\right]_{\partial_1 D} = \varphi_2; \quad (7b)$$

2.4. Based on [5, ch.2,3] and some of our latest works for the anisotropic inhomogeneous 2D nonlinear models of von Kármán-Mindlin-Reissner (KMR) type for binary mixture of porous, piezo-magneto-electric and electrically conductive and viscous elastic thin-walled structures with variable thickness and BVPs of Vekua type systems of DEs, we prove: i) there are truly Korn type inequalities, ii) operator factorized schemes(Rutishauser, Keldish, Gauss types) are constructed by means of which an approximate solution can be found with high order of accuracy; iii) by our modification of solution the projective method is stable.

3. Thus the solution of BVPs (1)-(2) by scheme (3) is reduced to the solutions of 1-dim BVPs. The approximate solution of such problems with corresponding smoothness by high order accuracy methods is possible [6-7]. We below underline principal sides of these methods considering typical BVP

$$y''(x) - q(x)y(x) = f(x),\ 0 < x < 1,\ (q \geq 0),\ y(0) = a,\ y(1) = b. \quad (8)$$

The method developed in [6] is reduced (8) to the three-point system of algebraic equations with a symmetric positive matrix. Coefficients of these algebraically analogy calculate by derivatives of given functions. This processes if we used formulas of numerical differentiation represent incorrect procedures.

Method from [7] for (8) also gives three-point recurrence relations with arbitrary order of accuracy. The coefficients of corresponding scheme defined as solutions of Volterra second kind integral equations. This method in theoretical sense looks like faultless. For real processes you must calculate multiple integrals by high order accuracy of quadrature formulas with more small mesh width than which was used for finding approximate solution of an initial problem.

The essential convenience of generalized factorization method by [8] is following: the difference from methods [6,7] is not present necessity to make up the table of multiple integrals or derivatives from initial data $q(x)$, $f(x)$.

The order of number arithmetical operation for calculating the coefficients and approximate solution of BVP (8) has first degree with respect to mesh width as in the classical cases.

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