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ILL-POSED PROBLEMS AND ASSOCIATED WITH THEM SPACES OF ORBITS AND ORBITAL OPERATORS

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Abstract. The ill-posed equation $Ku = f$ is considered, where $K : H \to H$ is a linear compact selfadjoint injective positive operator and H is a Hilbert space. The Hilbert space $D(K^{-n})$ of n-orbits of the operator K^{-1} is introduced taking into account the topology. We transfer the considered equation in this space and construct a linear central spline algorithm for approximate solution of transferred equation (Theorem 1). It is proved that the projective limit of the sequence of n-orbits spaces is the space of all orbits $D(K^{-\infty})$ in which the transferred equation becomes well posed.

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Let H be a separable Hilbert space with the norm $\|\cdot\|$, endowed with the inner product $(\cdot,\cdot)$, and let $K : H \to H$ be a compact, injective, selfadjoint, positive operator. The left inverse operator K^{-1} is not continuous. By an orbit of the operator K^{-1} at the point x a sequence $\text{orb}(K^{-1}, x) := \{x, K^{-1}x, \cdots, K^{-n}x, \cdots\}$ is defined. Under a n-orbit of K^{-1} at the point x we mean a sequence $\text{orb}_n(K^{-1}, x) := \{x, K^{-1}x, \cdots, K^{-n}x\},\ n \in Z_+$, where Z_+ is the set of nonnegative whole numbers. We denote by $D(K^{-\infty})$ the space of all orbits of K^{-1} and by $D(K^{-n})$ the space of all n-orbits of K^{-1}. It is obvious that $D(K^{-\infty})$ is a closed subspace of Frechet-Hilbert space H^N and $D(K^{-n})$ is subspace of Hilbert space H^{n+1} with the product topology.

When $K^{-1} = A$, the operator $A : H \to H$ is a positive definite, selfadjoint operator on H and the well-known Frechet space $D(A^\infty)$ coincides with the space $D(K^{-\infty})$. In [2] we have defined the operator A^∞ by equality, which, in the above notations, has the form

$$A^\infty(\text{orb}(A, x)) = \text{orb}(A, Ax).$$

In [3] we call A^∞ an orbital operator. Taking into account topologies of the Frechet space $D(A^\infty)$ and countable product H^N of Hilbert spaces H, we get that A^∞ coincides with the restriction of the operator $A^N : H^N \to H^N$ defined on H^N by equality $A^N\{x_n\} = \{Ax_n\}$ from the Frechet-Hilbert space H^N to $D(A^\infty)$. The operator is also defined

$$K^{-\infty}(\text{orb}(K^{-1}, u)) = \{K^{-1}u, K^{-2}u, \cdots, K^{-n}u, \cdots\} = \text{orb}(K^{-1}, K^{-1}u),$$

$K^{-\infty}$ is also called an orbital operator. It is known [2] that the operator $K^{-\infty}$ is continuous, positive definite and selfadjoint in the Frechet space $D(K^{-\infty})$ and admits the inverce one $(K^{-\infty})^{-1}$ which is also continuous in the Frechet space $D(K^{-\infty})$. Therefore, the operator $K^{-\infty}$ is a topological isomorphism of the Frechet space $D(K^{-\infty})$ onto itself. Let us denote the operator $(K^{-\infty})^{-1}$ by K_∞ i.e.

$$K_\infty(\text{orb}(K^{-1},u)) = \{Ku, u, K^{-1}u, \cdots\} = \text{orb}(K^{-1}, Ku).$$

It is easy to see that algebraically the restriction of the operator K on the set $D(K^{-\infty}) \subset H$ coincides with the operator K_∞. K_∞ is also a topological isomorphism of the Frechet space $D(K^{-\infty})$ onto itself and $K_\infty u = f$ admits unique and stable solution, i.e. this equation is well-posed in $D(K^{-\infty})$. We have studied the problem of approximate solution of the equation $K_\infty u = f$ in $D(K^{-\infty})$ and have constructed a linear, central spline algorithm [1].

We use the terminology and notations mainly from [5]. We will compute $S(f)$ for the solution operator of the equation $Au = f$.

Let $\{\varphi_k\}$ be an orthogonal sequence of eigenfunctions of the operator K with the corresponding sequence of eigenvalues $\{\lambda_k\}$, $k \in \mathbb{N}$. It is easy to see that $\{\varphi_k\}$ is a complete system in H. Then K has the form $Ku = \sum_{k=1}^{\infty} \lambda_k(\varphi_k,\varphi_k)^{-1}\ (u,\varphi_k)\varphi_k$, where $\lambda_k \to 0,\ \lambda_k > 0$.

The inverse of K^{-1} to the operator K is selfadjoint and has the form

$$K^{-1}x = \sum_{k=1}^{\infty} \lambda_k^{-1}(x,\varphi_k)(\varphi_k,\varphi_k)^{-1}\varphi_k\ .$$

The sequence λ_k^{-1} is unbounded and tends to infinity. Therefore, the selfadjoint operator K^{-1} has discrete spectrum ([6], p.98) and a dense image.

Let n be a fixed nonnegative whole number and let us consider elements of the space H to which we can apply the operator $K^{-n} = (K^{-1})^n$, where K^0 is the identite operator. It is possible to idetify every element $x \in D(K^{-n})$ with the n-orbit of the operator K^{-1} at the point x, i.e. with $\text{orb}_n(K^{-1},x)$. We hope that this identification will not cause misunderstandings. In such identification the set $D(K^{-n})$ is exactly the space of n-orbits of the operator K^{-n}. We can turn this set into a prehilbert space with the help of the following inner product

$$(x,y)_n = (x,y) + (K^{-1}x, K^{-1}y) + \cdots + (K^{-n}x, K^{-n}y),\ n \in Z_+. \tag{1}$$

According to (1), the norm of an element $x \in D(K^{-n})$ has the form

$$\|x\|_n = (\|x\|^2 + \|K^{-1}x\|^2 + \cdots + \|K^{-n}x\|^2)^{1/2},\ n \in Z_+. \tag{2}$$

It is easy to verify that if the operator K^{-1} is closed, then $D(K^{-n})$ is the Hilbert space.

Let us consider the equation

$$Ku = f, \tag{3}$$

in the space $D(K^{-n})$, which, in general, is not correct. In view of the foregoing, the equation (3) in the space $D(K^{-n})$ actually has the form

$$K_n(\text{orb}_n(K^{-1},u)) = \text{orb}_n(K^{-1},f), \tag{4}$$

where the operator $K_n : D(K^{-n}) \to D(K^{-n})$ is defined by the equality

$$K_n(\mathrm{orb}_n(K^{-1}, u)) = \mathrm{orb}_n(K^{-1}, Ku).$$

We will call K_n the n-orbital operator for the operator K.

It is well-known that the least squares solution u of the minimal norm of the operator equation $K_n u = f,\ f \in \mathrm{Im}K_n \cup (\mathrm{Im}K_n)^{\perp}$ is given by

$$u = \sum_{k=1}^{\infty} \lambda_k^{-1}(\varphi_k, \varphi_k)^{-1}(f, \varphi_k)\varphi_k,\ \ \lambda_k \to 0,\ \ \lambda_k > 0 \tag{5}$$

and the set of the least squares solutions coincides with the set of solutions of normal equaton $K_n^* K_n u = K_n^* f$.

The operator K_n is symmetric and positive in the space $D(K^{-n})$. The inverse operator K_n^{-1} has the form

$$K_n^{-1}(\mathrm{orb}_n(K^{-1}, x)) = \mathrm{orb}_n(K^{-1}, K^{-1}x) \tag{6}$$

and it is a symmetric and positive definite operator in the space $D(K^{-n})$.

Our goal is to build an algorithm for the approximate solution of the equation (4) in the space $D(K^{-n})$. For the construction of approximate solution $U(f)$ we apply some information about the problem element f . Let $y = I(f)$ be a nonadaptive computable information of the cardinality m, i.e.

$$y = I(f) = [L_1(f), \cdots, L_m(f)], \tag{7}$$

where $L_1, \cdots, L_m$ are linear functionals on the space H.

Let us construct an interpolating $y \in I(H)$ spline in the space $D(K^{-n})$. For this we consider the following spaces: the linear space F_1 consisting from elements of the space $D(K^{-n})$; $G = D(K^{-n})$ with the norm (2), the set of problem elements is $F = \{f \in F_1;\ ||T(f)||_n \leq 1\}$, where T is an identical operator from F_1 on $D(K^{-n})$ and $X = (D(K^{-n}), ||\cdot||_n)$. The solution operator S_n is K_n^{-1}, defined by equality (6). Let us assume that the information on $D(K^{-n})$ is given by (7), where $L_i(f) = (f, \varphi_i),\ i = 1, \cdots, m$. The interpolating $y = I(f)$ spline σ if defined by the equality $||T(\sigma)||_n = \inf\{||T(z)||_n,\ z \in I^{-1}(y)\}$. It has the form

$$\sigma_m(f) = \sigma_m(I(f)) = \sum_{k=1}^{m} \frac{(f, \varphi_k)}{(\varphi_k, \varphi_k)}\varphi_k \tag{8}$$

and is not depending on n.

To solve equation (4) by the Ritz method, it is necessary to consider the energetic space E_{K_n} of the operator K_n in the space $E = D(K^{-n})$. The inner product $[f_1, f_2]_n$ of elements f_1 f_2 of this energetic space is $(K_n f_1, f_2)_n$. The approximate solution obtained by Ritz method relatively to the system $\varphi_1, \varphi_2, \cdots, \varphi_m$, has the form

$$u_m = \sum_{k=1}^{m} \frac{(f, \varphi_k)}{(\varphi_k, \varphi_k)\lambda_k}\varphi_k = K_n^{-1} \sum_{k=1}^{m} \frac{(f, \varphi_k)}{(\varphi_k, \varphi_k)}\varphi_k = S_n \sigma_m. \tag{9}$$

This means that $U(f) = u_m$ is a spline algorithm. Analogously to ([5], p.97), we can prove also that the algorithm (9) is central.

Let us assume that the equation (4) admits a generalized solution with the finite energy u_0 in the energetic space E_{K_n}. It is proved in ([6], ch.5, §34) that the approximate solution u_m converges to u_0 in E_{K_n}. We have proved that the above notation is valid.

Theorem 1. *Let H be a Hilbert space, let K be a compact, injective selfadjoint, positive operator in H and let the operator K^{-1} be closed. We will still require that the set $T(\mathrm{Ker}I)$ is closed and the radius of information I is finite. Then the algorithm (9) is a linear central spline algorithm for the approximate solution of the equation $K_n u = f$ in the space $D(K^{-n})$. Besides, if in the energetic space of the operator K_n there exists the generalized solution u_0 with the finite energy, the sequence of the approximative solutions converges to u_0 in the energetic space E_{K_n}.*

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