Reports of Enlarged Sessions of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 32, 2018

FINITE-DIFFERENCE SCHEME FOR ONE-DIMENSIONAL ANALOG OF THE MAXWELL NONLINEAR SYSTEM WITH SOURCE TERM

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Abstract. One-dimensional analog of the Maxwell nonlinear system with source term is considered. The finite-difference scheme for the numerical solution of the posed initial-boundary value problem is constructed. Convergence of that scheme is given.

Keywords and phrases: One-dimensional nonlinear Maxwell equations, finite-difference scheme, convergence.

AMS subject classification (2010): 35Q60, 35K55, 65M06.

Construction, investigation, and computer realization of algorithms for approximate solution of problems describing applied processes represent the actual sphere of mathematics.

Various mathematical models of diffusive processes lead to nonstationary partial differential systems of equations. Most of those problems, as a rule are nonlinear. This moment significantly complicates the investigation of such models.

It is well known that the process of penetration of the electromagnetic field into a medium is described by the nonlinear partial differential system of the Maxwell equations [1]. The main characteristic of this system is that it contain equations, which are strongly connected to each other. This circumstance dictates to use the corresponding investigation methods for each concrete model, as the general theory for such systems is not yet developed. Naturally, the questions of numerical solution of these problems, which also are connected with serious complexities, arise as well. Based on this model, on the rectangle $[0,1]\times[0,T]$ consider the following first type initial-boundary value problem for one-dimensional system with the source term $|U|^{q-2}U$, $q\geq 2$:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V\frac{\partial U}{\partial x}\right)-|U|^{q-2}U, \tag{1}$$

$$\frac{\partial V}{\partial t}=\left(\frac{\partial U}{\partial x}\right)^2, \tag{2}$$

$$U(t,0)=U(t,1)=0, \tag{3}$$

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x)\geq Const>0, \tag{4}$$

where U_0,V_0 are known functions of their arguments and T is the fixed positive constant.

System of equations (1), (2) is interesting for mathematics, as well as and other scientific fields.

It is well known that system (1), (2) describes many other physical processes (see, for example, [2] - [7] and references therein).

Note that system (1), (2) can be reduced to the integro-differential form. Many works are devoted to the integro-differential models of that type (see, for example, [10] and references therein).

The questions of existence, uniqueness, long time behavior of the solutions and numerical resolution of some kind of initial-boundary value problems for above-mentioned integro-differential models are studied in many works (see, once again, [10] and references therein).

It is important to construct and study discrete analogs for the model (1), (2). The semi-discrete and finite difference second order accuracy schemes with respect of spatial step is constructed and studied in [8] for (1) - (4) problem without source term $|U|^{q-2}U$ in (1).

In [9] more general finite difference schemes including second order accuracy two- and three-level type schemes are also studied for this model. The same questions are also investigated in [10].

Our aim in this note is to generate these type investigations for (1) - (4) problem with source term in equation (1). Let us introduce on the rectangle $[0,1]\times[0,T]$ the grids:

$$\omega_{h\tau}=\bar{\omega}_h\times\omega_\tau,\quad \omega^*_{h\tau}=\bar{\omega}^*_h\times\omega_\tau,$$

where

$$\omega_\tau=\{t_j=j\tau,\quad j=0,1,...,N,\quad \tau=T/N\},$$

$$\bar{\omega}_h=\{x_i=ih,\quad i=0,1,...,M,\quad h=1/M\},\quad \omega_h=\bar{\omega}_h\setminus\{x_0,x_M\},$$

$$\omega^*_h=\{x^*_i=(i-1/2)h,\quad i=1,2,...,M\}.$$

Let us introduce also well-known scalar-products, norms and notations [11]:

$$(y,z)=\sum_{i=1}^{M-1}y_iz_ih,\quad (y,z]=\sum_{i=1}^{M}y_iz_ih,\quad \|y\|=(y,y)^{1/2},\quad \|y]|=(y,y]^{1/2},$$

$$y_x=\frac{y_{i+1}-y_i}{h},\quad y_{\bar{x}}=\frac{y_i-y_{i-1}}{h},\quad y_t=\frac{y^{j+1}-y^j}{\tau},\quad y_{\bar{t}t}=\frac{y^{j+1}-2y^j+y^{j-1}}{\tau^2},$$

$$\hat{y}=y^{j+1},\quad y^{(\sigma)}=\sigma\hat{y}+(1-\sigma)y$$

and for problem (1) - (4) consider the following finite-difference scheme:

$$u_t + \mu\tau u_{\bar{t}t} = (v^{(\sigma)} u_{\bar{x}}^{(\sigma)})_x - |\hat{u}|^{q-2}\hat{u}, \tag{5}$$

$$v_t + \mu\tau v_{\bar{t}t} = (u_{\bar{x}}^{(\sigma)})^2, \tag{6}$$

$$u(0,t) = u(1,t) = 0, \quad t \in \omega_\tau, \tag{7}$$

$$u(x,0) = U_0(x), \quad x \in \omega_h, \quad v(x,0) = V_0(x), \quad x \in \omega_h^* \tag{8}$$

$$\begin{aligned} u(x,\tau) &= U_0(x) + \tau \left(VU_{\bar{x}}\right)_x\Big|_{t=0}, \quad x \in \omega_h, \\ v(x,\tau) &= V_0(x) + \tau \left(U_{\bar{x}}\right)^2\Big|_{t=0}, \quad x \in \omega_h^*. \end{aligned} \tag{9}$$

In (5), (6) the discrete function u is defined on $\omega_{h\tau}$ and v is defined on $\omega_{h\tau}^*$.

The following statement takes place.

Theorem. *If $q \geq 2$, $\sigma - 0.5 \geq \mu \geq 0$ and problem (1) - (4) has a sufficiently smooth solution, then finite difference scheme (5) - (9) converges as $\tau \to 0$, $h \to 0$, and the following estimate is true*

$$\left\|U^j - u^j\right\| + \left\|V^j - v^j]\right| = O(\tau^2 + h^2 + (\sigma - 0.5 - \mu)\tau).$$

It is clear that from Theorem we get the following result: if $\sigma = 0.5$, $\mu = 0$ or $\sigma = 1$, $\mu = 0.5$ then convergence is the second order $O\left(\tau^2 + h^2\right)$.

Various numerical experiments using above mentioned finite-difference scheme (5) - (9) and well-known iterative methods [12] are carried out. The results of these experiments agree with theoretical investigations.

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Received 20.05.2018; revised 23.10.2018; accepted 13.12.2018.

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