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ROBUST STOCHASTIC CONTROL OF THE EXCHANGE RATE USING INTEREST RATES

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Abstract. We consider the problem of a Central Bank that wants the exchange rate to be as close as possible to a given target, and in order to do that uses the interest rate level in the foreign exchange market. We model this as a robust stochastic control problem, and provide Bellman-Isaacs equation to that kind of problem.

Keywords and phrases: Exchange rate, Bellman-Isaacs equation, robust stochastic control.

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Let us consider a probability space (Ω, F, P) together with a filtration (F_t) generated by a one-dimensional Brownian motion W, that represents the uncertainty in the economy. We consider a given foreign currency and a domestic currency and we denote the exchange rate by X_t :=domestic currency units per unit of foreign currency at time t. We assume that X is an adapted stochastic process given by

$$X_t = x + \int_0^t (\mu_s X_s + K u_s) ds + \int_0^t \sigma_s X_s dW_s. \tag{1}$$

Here, $\mu_s \in [\mu_-, \mu_+]$ is the exogenous economic pressure on the exchange rate level. The constant $K \in (-\infty, 0]$ is the influence of the interest rate level on the exchange rate level, and the stochastic process $\sigma_s \in [\sigma_-, \sigma_+], \sigma_- > 0$ is the exogenous volatility in the exchange rate. The stochastic process $r(t)$ is the domestic interest rate, and the constant $\bar{r}$ is its target. $u(t) := \log(r(t)/\bar{r})$. The stochastic process u measures the relationship between the interest rate level $r(t)$ set by the Central Bank and the target $\bar{r}$. Since we consider the logarithm of this ratio, when the interest rate level r is above the target $\bar{r}$, u will be positive and, therefore, the interest rate will push X downward and the exchange rate will tend to appreciate; when r is lower than $\bar{r}$, the opposite occurs. It is rational that $\bar{r}$ is perceived as a natural long-term equilibrium interest rate and deviations from it will make domestic securities relatively more or less attractive. Furthermore, by taking the logarithm of the ratio, we make sure that the relevant value is the proportion of the change in the ratio. We observe that, in the particular case in which the interest rate does not affect the exchange rate, X is just a geometric Brownian motion.

Definition 1. A stochastic control $u : [0, \infty) \times \Omega \to R$ is an (F_t)-adapted stochastic process. We consider only those stochastic controls such that $P(\forall t \in [0, \infty) : X(t) \in [0, \infty)) = 1$.

The Central Bank wants to select u that minimizes the functional J defined by

$$J(x;u) := \max_{(\mu,\sigma)\in[\mu_-,\mu_+]\times[\sigma_-,\sigma_+]} E\bigg[\int_0^\infty e^{-\lambda t} f(X_t,u_t)dt\bigg],$$

where

$$f(x,u) = (x-\rho)^2 + ku^2,$$

which is optimal stochastic control of the exchange rate. Here, f represents the running cost incurred by deviating from the aimed targets both for the exchange rate and the interest rate: ρ is the exchange rate target, $\bar{r}$ is the interest rate target, and k is a normalization positive constant.

Since we want to minimize the functional J, we should consider only those strategies for which J is finite. We claim

$$E\int^\infty e^{-\lambda t} X^2(t)dt < \infty, \tag{2}$$

$$E\int_0^\infty e^{-\lambda t} u^2(t)dt < \infty, \tag{3}$$

$$\lim_{T\to\infty} Ee^{-\lambda T}X(T) = 0. \tag{4}$$

Definition 2. (Admissible controls). We shall say that a stochastic control u is admissible conditions (2)-(4) are satisfied. We shall denote by $\mathcal{A}(x)$ the class of admissible stochastic controls.

Let us denote by V the value function. That is, for every $x\in(0,\infty)$,

$$V(x) := \inf\{J(x;u); u\in\mathcal{A}(x)\}.$$

Let us consider the operator $\mathcal{L}^u$ defined by

$$\mathcal{L}^u\psi(x) := \max_{(\mu,\sigma)\in[\mu_-,\mu_+]\times[\sigma_-,\sigma_+]}\left[\frac{1}{2}x^2\sigma^2\psi''(x) + x\mu\psi'(x)\right] + Ku\psi'(x) - \lambda\psi(x)$$
$$= \frac{1}{2}x^2(\sigma_+^2\psi''(x)^+ - \sigma_-^2\psi''(x)^-) + x(\mu_+\psi'(x)^+ - \mu_-\psi'(x)^-) + Ku\psi'(x) - \lambda\psi(x).$$

Now we intend to characterize the value function and an associated optimal strategy.

The Bellman-Isaacs equation

$$\min_{u\in(-\infty,\infty)}\{\mathcal{L}^u v(x) + f(x,u)\} = 0$$

can be written as

$$\begin{aligned}\frac{1}{2}x^2\big(\sigma_+^2V''(x)^+ - \sigma_-^2V''(x)^-\big) + x\big(\mu_+V'(x)^+ - \mu_-V'(x)^-\big)\\ -\frac{K^2}{4k}V'(x)^2 - \lambda V(x) + (x-\rho)^2 = 0.\end{aligned} \tag{5}$$

Since $p \to \frac{1}{\sigma_+^2}p^+ - \frac{1}{\sigma_-^2}p^-$ is the inverse of the function $p \to \sigma_+^2 p^+ - \sigma_-^2 p^-$ then (5) is equivalent to

$$\begin{aligned} &\frac{1}{2}V''(x) + \frac{1}{\sigma_+}\Big[\frac{1}{x}(\mu_+ V'(x)^+ - \mu_- V'(x)^-) \\ &-\frac{K^2}{4kx^2}V'(x)^2 - \frac{\lambda}{x^2}V(x) + \frac{1}{x^2}(x-\rho)^2\Big]^+ \\ &-\frac{1}{\sigma_-}\Big[\frac{1}{x}(\mu_+ V'(x)^+ - \mu_- V'(x)^-) \\ &-\frac{K^2}{4kx^2}V'(x)^2 - \frac{\lambda}{x^2}V(x) + \frac{1}{x^2}(x-\rho)^2\Big]^- = 0. \end{aligned} \tag{6}$$

From the solution of (5) it is possible to construct the optimal stochastic control $\hat{u}$ by

$$\hat{u}(t) = -\frac{K}{2k}V'(\hat{X}(t)),$$

or, equivalently, the optimal interest rate would be given by

$$\hat{r}(t) = \bar{r}e^{-\frac{K}{2k}V'(\hat{X}(t))},$$

where $\hat{X}$ is the solution of equation (1) corresponding to the optimal control $\hat{u}$.
Remark. If $\sigma_- = \sigma_+ = \sigma$, $\mu_- = \mu_+ = \mu$, then equation (5) becomes

$$\frac{1}{2}x^2\sigma^2 V''(x) + x\mu V'(x) - \frac{K^2}{4k}V'(x)^2 - \lambda V(x) + (x-\rho)^2 = 0$$

and admits solution of the form $V(x) = Ax^2 + 2Bx + C$, with some constants, which is unique in the class of functions with quadratic growth. This case was considered in [1], [2], however authors consider the problem of a Central Bank to control optimally the exchange rate by using two tools: direct intervention in the foreign exchange market and indirect intervention through determination of the interest rate levels.

By the standard technics of dynamic programming principles (as in [3]) we prove

Theorem. *Let V be the classical solution of quadratic growth of (5). Then V coincides with value of the minmax problem*

$$\min_u J(x;u) := \min_u \max_{(\mu,\sigma)\in[\mu_-,\mu_+]\times[\sigma_-,\sigma_+]} E\int_0^\infty e^{-\lambda t} f(X_t, u_t)dt$$

and optimal strategies are given by

$$\hat{u}(t) = -\frac{K}{2k}V'(\hat{X}(t)),$$

$$\hat{\mu}(t) = \mu_+ I_{(V'(\hat{X}(t))>0)} + \mu_- I_{(V'(\hat{X}(t))<0)},$$
$$\hat{\sigma}(t) = \sigma_+ I_{(V''(\hat{X}(t))>0)} + \sigma_- I_{(V''(\hat{X}(t))<0)}.$$

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